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Stability, Stabilization and Observers of Linear Control Systems on Time Scales

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***Abstract*— Linear control systems defined on arbitrary time scales are studied. It is shown that the classical results on stabilization and detectability for linear continuous-time and discrete-time systems can be extended to systems on arbitrary time scales. These results depend on the exponential stability criteria, which are different for different time scales. The set of exponential stability, which appears in these criteria, is studied. It is shown that it may be empty, which leads to some pathologies in the system behavior.**

***Index Terms*— exponential stability, observers, stabilization, time scales**

I. INTRODUCTION AND MOTIVATION

Exponential stability for continuous-time and discrete-time linear systems with constant coefficients are characterized by different criteria: the spectrum of the matrix of coefficients must lie in the stability set which is either the left open half-plane or the unit open disc with the center at 0, respectively (see e.g. [1]). If the standard description of the discrete-time system

$$x(k+1) = Ax(k)$$

is changed to the difference form

$$x^{\Delta}(k) := x(k+1) - x(k) = \tilde{A}x(k),$$

where $\tilde{A} = A - I$, the eigenvalues of exponentially stable matrix $\tilde{A}$ lie now in the unit open disc with the center at -1. This is easily generalized to systems of the form

$$(x(kh+h) - x(kh))/h = \tilde{A}x(kh),$$

where $k \in \mathbb{Z}$. The stability set is now the open disc of the radius $1/h$ and the center at $-1/h$. Observe that when h tends to 0, this disc becomes the left open halfplane.

All these models of time are particular cases of time scales, which are arbitrary closed subsets of the set of real numbers. The sets $\mathbb{R}$ and $h\mathbb{Z}$ for $h > 0$ are homogeneous time scales. However in many models, time does not have to be homogeneous. A union of disjoint closed intervals is an example of a nonhomogeneous time scale. Here time is partly continuous and partly discrete. One can refer to [2] for examples of concrete systems evolving on such time scales. The calculus on time scales, whose foundations were created by Stefan Hilger [3] in 1988, allows to use a common language for continuous-time and discrete-time dynamical systems. In fact, we can study dynamical systems on an arbitrary time scale employing so called delta derivative. Basic properties of linear control systems on arbitrary time scales were examined by Bartosiewicz and Pawłuszewicz [4] and Fausett and Murty [5]. Baylor Time Scales Research Group is working on both theoretical and practical aspects of time-scales models. In particular, they use time-scales language for description of hybrid systems (see e.g. [6], [7]).

Exponential stability of systems on nonhomogeneous time scales is more complicated. It was first studied by Aulbach and Hilger [8], but the most complete results were obtained by Pötzsche, Siegmund and Wirth [9]. They defined the set of exponential stability for an arbitrary time scale. This set may be quite complicated, may be very small or even empty. We give examples of time scales for which no exponentially stable system exist. Such time scales may appear as the effect of nonhomogeneous discretization of time. As we are interested in stabilization and detectability of linear systems, existence of exponentially stable systems for a given time scale is of practical importance.

Once it is guaranteed that the set of exponential stability is nonempty, we can extend the results on stabilization and detectability known for continuous-time systems to systems on arbitrary time scales. The main feature here is the fact that controllability and observability criteria are the same for all the time scales and they depend only on the matrices of the system (see [4], [5], [10]). Also standard constructions of observers may be extended to systems on arbitrary time scales with nonempty sets of exponential stability.

II. CALCULUS ON TIME SCALES

We recall here basic concepts and facts of the calculus on time scales. For more information the reader is referred to [2].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. The standard examples of time scales are $\mathbb{R}$, $h\mathbb{Z}$, $h > 0$, $\mathbb{N}$, $\mathbb{N}_0$, $2^{\mathbb{N}_0}$ or $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty}[k(a+b), k(a+b)+a]$. The time scales $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$.

The following operators on $\mathbb{T}$ are often used:

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\sigma(\sup\mathbb{T}) = \sup\mathbb{T}$, if $\sup\mathbb{T} \in \mathbb{T}$,
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$, defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ and $\rho(\inf\mathbb{T}) = \inf\mathbb{T}$, if $\inf\mathbb{T} \in \mathbb{T}$,

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- the *graininess function* $\mu : \mathbb{T} \to [0,\infty)$ defined by $\mu(t) := \sigma(t) - t$.

Points from the time scale can be classified as follows: a point $t \in \mathbb{T}$ is called

- *right-scattered* if $\sigma(t) > t$ and *right-dense* if $\sigma(t) = t$,
- *left-scattered* if $\rho(t) < t$ and *left-dense* if $\rho(t) = t$.

We define also the set

$$\mathbb{T}^{\kappa} := \begin{cases} \mathbb{T} \setminus (\rho(\sup\mathbb{T}), \sup\mathbb{T}], & \text{if } \sup\mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup\mathbb{T} = \infty. \end{cases}$$

Definition 2.1: Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^{\kappa}$. Then the number $f^{\Delta}(t)$ (when it exists), with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \ \forall s \in U,$$

is called the *delta derivative of f at t*. The function $f^{\Delta} : \mathbb{T}^{\kappa} \to \mathbb{R}$ is called the *delta derivative* of f on $\mathbb{T}^{\kappa}$. We say that f is *delta differentiable* on $\mathbb{T}^{\kappa}$, if $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^{\kappa}$.

Remark 2.2:

- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff $f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t)$, i.e. f is differentiable in the ordinary sense at t.
- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable on $\mathbb{Z}$ and $f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t)$, for all $t \in \mathbb{Z}$.
- If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $f : \mathbb{T} \to \mathbb{R}$ is always delta differentiable on $\mathbb{T} \setminus \{0\}$ and $f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}$, for all $t \in q^{\mathbb{Z}}$. Moreover, $f^{\Delta}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}$, if only this limit exists.

Definition 2.3: A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* if its right-side limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at all left-dense points in $\mathbb{T}$.

Definition 2.4: A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 2.5: A continuous function $f : \mathbb{T} \to \mathbb{R}$ is *pre-differentiable* with D (the region of differentiation), if $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$ and f is differentiable at each $t \in D$.

If f is regulated then there is a function F that is pre-differentiable with the region of differentiation D such that $F^{\Delta}(t) = f(t)$, for all $t \in D$. Any function F that satisfies $F^{\Delta}(t) = f(t)$ is called a *pre-antiderivative* of f. Then the *indefinite integral* of a regulated function f is defined by $\int f(t)\Delta t = F(t) + C$, where C is arbitrary constant. The *Cauchy integral* of a regulated function f is defined by $\int_r^s f(t)\Delta t = F(s) - F(r)$, for all $s, t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ if it satisfies $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^{\kappa}$.

It is known that every rd-continuous function has an antiderivative.

Example 2.6:

- If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right is the usual Riemann integral.
- If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then $\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h \cdot f(kh)$, for $a < b$.

III. Exponential stability

Let $\sup\mathbb{T} = \infty$ and $A \in \mathbb{C}^{n \times n}$ be a constant matrix.

Let us consider the following linear system of delta differential equations on time scales $\mathbb{T}$

$$x^{\Delta}(t) = Ax(t), \tag{1}$$

where $t \in \mathbb{T}$ and $x(t) \in \mathbb{C}^n$.

Theorem 3.1: [2] Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{C}^n$. Then the system (1) with the initial condition $x(t_0) = x_0$ has a unique solution $x : [t_0, +\infty) \cap \mathbb{T} \to \mathbb{C}^n$.

This result can be extended to matrix-valued solutions of (1), which leads to the following definition.

Definition 3.2: Let $t_0 \in \mathbb{T}$. A function $X : [t_0, +\infty) \to \mathbb{C}^{n \times n}$ that satisfies the matrix delta differential equation

$$X^{\Delta}(t) = AX(t), \tag{2}$$

and the initial condition $X(t_0) = I$, where I is the $n \times n$ identity matrix, is called *the matrix exponential function* (corresponding to A) *initialized at* t_0. Its value at $t \in \mathbb{T}$, $t \geqslant t_0$, is denoted by $e_A(t, t_0)$.

Example 3.3:

- If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = \mathrm{e}^{A(t-t_0)}$, where $t, t_0 \in \mathbb{R}$.
- If $\mathbb{T} = h\mathbb{Z}$, then $e_A(t, t_0) = (I + hA)^{\frac{t-t_0}{h}}$, where $t, t_0 \in h\mathbb{Z}$, $t \geqslant t_0$.
- If $\mathbb{T} = 2^{\mathbb{N}_0}$, then $e_A(t, t_0) = \prod_{k=1}^{\log_2 \frac{t}{t_0}} \left(I + \frac{t}{2^k}A\right)$, where $t, t_0 \in \mathbb{T}$, $t \geqslant t_0$.

Remark 3.4: The vector function $t \mapsto e_A(t, t_0)x_0$ is the solution of (1) with the initial condition $x(t_0) = x_0$.

Definition 3.5: The system (1) is *exponentially stable* if there exists a constant $\alpha > 0$ such that for every $t_0 \in \mathbb{T}$ there exists $K = K(t_0) \geqslant 1$ with

$$\| e_A(t, t_0)x(t_0) \| \leqslant K\mathrm{e}^{-\alpha(t-t_0)} \| x(t_0) \| \tag{3}$$

for $t \geqslant t_0$.

We will be mostly concerned with real matrices A. For the initial state from $\mathbb{R}^n$, the solution x takes on the values from $\mathbb{R}^n$ as well.

The following example shows that there are time scales for which no system can be exponentially stable.

Example 3.6: Assume $\mathbb{T} = 2^{\mathbb{N}_0}$, $n = 1$ and $A = a \in \mathbb{C}$. Let $k_0 \in \mathbb{N}_0$ be such that for all $k \geqslant k_0$, $|1 + a2^k| \geqslant 1$. Then for $t_0 = 2^{k_0} \in \mathbb{T}$ the solution of (1) on that time scale is equal

$$x(2^k) = (1 + 2^{k-1}a)(1 + 2^{k-2}a)\dots(1 + 2^{k_0}a)x(2^{k_0}).$$

For $x(2^{k_0}) \neq 0$ it does not convergence to zero when k goes to infinity. Hence, according to Definition 3.5, (1) is not exponentially stable. We shall see later that for $n>1$ the situation is the same.

The following theorem characterizes the exponential stability of a scalar equation, i.e. for $n=1$.

Theorem 3.7: [9] Let $\lambda \in \mathbb{C}$. The scalar equation

$$x^{\Delta}(t)=\lambda x(t) \tag{4}$$

is exponentially stable if and only if one of the following conditions is satisfied:

(i) for arbitrary $t_0 \in \mathbb{T}$ $\gamma(\lambda):=\limsup_{T\to\infty}\frac{1}{T-t_0}\int_{t_0}^{T}\lim_{s\to\mu(t)}\frac{\log|1+s\lambda|}{s}\Delta t<0,$

(ii) $\forall T\in\mathbb{T}\ \ \exists t\in\mathbb{T}$ with $t>T$ such that $1+\mu(t)\lambda=0$,

where we use the convention $\log 0 = -\infty$ in (i).

From conditions (i) and (ii) in Theorem 3.7 we can define sets that contain all $\lambda\in\mathbb{C}$ for which (4) is exponentially stable.

Definition 3.8: [9] For a fixed $t_0\in\mathbb{T}$ we define

$$S_{\mathbb{C}}(\mathbb{T}):=\{\lambda\in\mathbb{C}:\limsup_{T\to\infty}\frac{1}{T-t_0}\int_{t_0}^{T}\lim_{s\to\mu(t)}\frac{\log|1+s\lambda|}{s}\Delta t<0\}$$

and

$$S_{\mathbb{R}}(\mathbb{T}):=\{\lambda\in\mathbb{R}:\ \forall T\in\mathbb{T}\ \ \exists t\in\mathbb{T},\ t>T\ :\ 1+\mu(t)\lambda=0\}.$$

Then the set defined by

$$S(\mathbb{T}):=S_{\mathbb{C}}(\mathbb{T})\cup S_{\mathbb{R}}(\mathbb{T})$$

is called the *set of exponential stability*.

Remark 3.9: The set $S_{\mathbb{C}}(\mathbb{T})$ does not depend on t_0. Additionally for an arbitrary time scale $\mathbb{T}$ we have $S_{\mathbb{C}}(\mathbb{T})\subseteq\{\lambda\in\mathbb{C}:\ Re\lambda<0\}$ and $S_{\mathbb{R}}(\mathbb{T})\subset(-\infty,0)$.

Example 3.10:

- Let $\mathbb{T}=\mathbb{R}$. Then $S_{\mathbb{R}}(\mathbb{R})=\emptyset$ and $S(\mathbb{R})=\{\lambda\in\mathbb{C}:\ Re\lambda<0\}$.
- Let $\mathbb{T}=h\mathbb{Z}$, where $h>0$. Then $S_{\mathbb{R}}(h\mathbb{Z})=\left\{-\frac{1}{h}\right\}$ and $S(h\mathbb{Z})=\mathscr{B}_{\frac{1}{h}}\left(-\frac{1}{h}\right)$, where $\mathscr{B}_{\frac{1}{h}}\left(-\frac{1}{h}\right)$ denotes the disc with the center at $\left(-\frac{1}{h},0\right)$ and the radius of $\frac{1}{h}$.
- Let $\mathbb{T}=\mathbb{H}=\{t_n\}$, where $t_n:=\sum_{k=1}^{n}\frac{1}{k}$, $n\in\mathbb{N}$. Then $S_{\mathbb{R}}(\mathbb{H})=\emptyset$ and $S_{\mathbb{C}}(\mathbb{H})=\{\lambda\in\mathbb{C}:\ Re\lambda<0\}$.
- Let $\mathbb{T}=\mathbb{P}_{a,b}=\bigcup_{k=0}^{\infty}[k(a+b),k(a+b)+a]$ be a union of closed intervals, where $a,b>0$. Then

$$\int_{k(a+b)}^{(k+1)(a+b)}\lim_{s\to\mu(t)}\frac{\log|1+\mu(t)\lambda|}{\mu(t)}\Delta t=\int_{k(a+b)}^{k(a+b)+a}Re\lambda dt+$$
$$+\int_{k(a+b)+a}^{(k+1)(a+b)}\frac{\log|1+b\lambda|}{b}\Delta t=aRe\lambda+\log|1+b\lambda|.$$

Because for every $t_0\in\mathbb{P}_{a,b}$ and for all $k\in\mathbb{N}_0$ we have $t_0+k(a+b)\in\mathbb{P}_{a,b}$ so we can use Lemma 9(b) from [9] and we get $S_{\mathbb{C}}(\mathbb{P}_{a,b})=\{\lambda\in\mathbb{C}:\ aRe\lambda+\log|1+b\lambda|<0\}$. Moreover $S_{\mathbb{R}}(\mathbb{P}_{a,b})=\left\{-\frac{1}{b}\right\}$.

On the basis of the above examples we see that the sets of exponential stability depend on time scales. There are time scales for which the set $S(\mathbb{T})$ is empty.

Example 3.11: Let $\mathbb{T}=q^{\mathbb{N}_0}=\{1,q,q^2,\ldots\}$, where $q>1$. Because

$$\lim_{t\to\infty}\lim_{s\to\mu(t)}\frac{\log|1+s\lambda|}{s}\Delta s=\lim_{t\to\infty}\frac{\log|1+(q-1)t\lambda|}{(q-1)t}=0$$

so from Lemma 9(a) of [9] we obtain $S_{\mathbb{C}}(q^{\mathbb{N}_0})=\emptyset$. It is also clear that $S_{\mathbb{R}}(q^{\mathbb{N}_0})=\emptyset$.

Proposition 3.12: If the graininess function is bounded then $S_{\mathbb{C}}(\mathbb{T})\neq\emptyset$.

Proof: Assume that $\mu(t)<\mu_0$ for all $t\in\mathbb{T}$. Then there is $\lambda\in\mathbb{R}$ such that $-\frac{1}{\mu_0}<\lambda<0$. Thus $0<1+\mu(t)\cdot\lambda<1$ and $\frac{\log|1+\mu(t)\lambda|}{\mu(t)}<0$, for all $t\in\mathbb{T}$ and $\mu(t)>0$.

If $\mu(t)=0$, then $\lim_{s\to\mu(t)}\frac{\log|1+s\lambda|}{s}=\lambda<0$, so $\gamma(\lambda)<0$. Hence $\lambda\in S_{\mathbb{C}}(\mathbb{T})$, so $S_{\mathbb{C}}(\mathbb{T})\neq\emptyset$. ■

Proposition 3.13: If $S_{\mathbb{C}}(\mathbb{T})\neq\emptyset$, then $S_{\mathbb{C}}(\mathbb{T})$ is infinite and $S(\mathbb{T})$ is symmetric with respect to the real axis.

Proof: From Definition 3.5, $S_{\mathbb{C}}$ is an open subset of $\mathbb{C}$. Hence $S_{\mathbb{C}}(\mathbb{T})$ is infinite. Furthermore $|1+\lambda s|=|1+\bar{\lambda}s|$ for real s, so $S_{\mathbb{C}}(\mathbb{T})$ is symmetric with respect to the real axis. This is also true for $S_{\mathbb{R}}(\mathbb{T})$. ■

Corollary 3.14: If $S_{\mathbb{C}}(\mathbb{T})\neq\emptyset$, then for all $n\in\mathbb{N}$ there exist $\alpha_0,\alpha_1,\ldots,\alpha_{n-1}\in\mathbb{R}$ such that the roots of $w(\lambda)=\lambda^n+\alpha_0\lambda^{n-1}+\ldots+\alpha_{n-1}$ are distinct and lie in $S_{\mathbb{C}}(\mathbb{T})$.

From Proposition 3.12 the boundedness of the graininess function is a sufficient condition for $S(\mathbb{T})$ to be nonempty. However, it is not a necessary condition, as there exist time scales with unbounded graininess function for which $S(\mathbb{T})$ is nonempty.

Example 3.15: Let $\mathbb{T}=\bigcup_{k=1}^{\infty}[t_k,t'_k]$, where $t_k\in\mathbb{R}$, $k\in\mathbb{N}$, $t_k<t'_k<t_{k+1}$. Define $\Delta_k:=t'_k-t_k$ and $\mu_k:=t_{k+1}-t'_k$. Assume $\mu_k\leqslant\Delta_k$ and $\lim_{k\to\infty}\mu_k=\infty$. Thus the graininess function is unbounded.

Let us consider the initial value problem

$$x^{\Delta}(t)=ax(t),\ x(t_0)=x_0,\ t\in\mathbb{T},\ t_0\in[t_k,t'_k], \tag{5}$$

where $a\in\mathbb{R}$, $a<0$. Then for $t\in[t_n,t'_n]$

$$x(t)=\mathrm{e}^{a(t'_k-t_0)}(1+a\mu_k)(\prod_{i=k+1}^{n-1}(1+a\mu_i)\mathrm{e}^{a\Delta_i})\mathrm{e}^{a(t-t_n)}x_0$$

is the solution of (5). If there is $i\geqslant k$ such that $\mu_i=-\frac{1}{a}$, then $x(t)\equiv 0$ from some moment.

Assume that $\mu_i\neq-\frac{1}{a}$, for all $i\geqslant k$. Since $a<0$, $|1+a\mu_i|\leqslant$

$e^{-\frac{1}{2}a\mu_i}$, for all $\mu_i \geqslant 0$. Then

$$\begin{aligned} |x(t)| &= e^{a(t'_k - t_0)}|1+a\mu_k|(\prod_{i=k+1}^{n-1}|1+a\mu_i|e^{a\Delta_i})e^{a(t-t_n)}|x_0| \leqslant \\ &\leqslant e^{a(t'_k - t_0)}|1+a\mu_k|(\prod_{i=k+1}^{n-1}|1+a\mu_i|e^{a\mu_i})e^{a(t-t_n)}|x_0| \leqslant \\ &\leqslant e^{a(t'_k - t_0)}e^{-\frac{1}{2}a\mu_k}(\prod_{i=k+1}^{n-1}e^{\frac{1}{2}a\mu_i})e^{a(t-t_n)}|x_0| = \\ &= e^{a(t-t_0)}e^{a(t'_k-t_n)}|x_0|e^{\frac{1}{2}a\left(\sum_{i=k+1}^{n-1}\mu_i - \mu_k\right)} = \\ &= e^{a(t-t_0+t'_k-t_n)}|x_0|e^{\frac{1}{2}a\left(\sum_{i=k+1}^{n-1}\mu_i - \mu_k\right)} \leqslant \\ &\leqslant |x_0|e^{\frac{1}{2}a\left(\sum_{i=k+1}^{n-1}\mu_i - \mu_k\right)}. \end{aligned}$$

Because $\lim_{k\to\infty}\mu_k = \infty$, so $\lim_{t\to\infty}x(t) = 0$. Hence the considered system is exponentially stable.

Remark 3.16: If the graininess function is increasing, then $S_{\mathbb{R}}(\mathbb{T}) = \emptyset$.

Let $\sigma(A)$ denote the set of all eigenvalues of the matrix A. The following theorem characterizes the exponential stability of (1).

Theorem 3.17: [9] The following assertions hold:

- If (1) is exponentially stable, then $\sigma(A) \subset S(\mathbb{T})$.
- If A is diagonalizable, then the system (1) is exponentially stable if and only if $\sigma(A) \subset S(\mathbb{T})$.

Example 3.18: Let us consider the following linear dynamic system

$$x^{\Delta} = \begin{pmatrix} -5 & -4 \\ 2 & 1 \end{pmatrix} x, \qquad (6)$$

which is defined on some time scale $\mathbb{T}$. The eigenvalues of the coefficient matrix A in (6) are $\lambda_1 = -1$ and $\lambda_2 = -3$, so the matrix is diagonalizable.

Let us take $\mathbb{T} = \mathbb{R}$. Then $S(\mathbb{T})$ consists of those λ for which $Re\lambda < 0$, so $\sigma(A) \subset S(\mathbb{T})$ and the system is exponentially stable.

If we take $\mathbb{T} = 2\mathbb{Z}$ and $t_0 = 0$, then the stability set is equal $\mathfrak{B}_{\frac{1}{2}}\left(-\frac{1}{2}\right)$ and it does not contain the eigenvalues of A. Thus the system is not exponentially stable for $\mathbb{T} = 2\mathbb{Z}$.

Let now $\mathbb{T} = \frac{1}{2}\mathbb{Z}$. In this case the eigenvalues of A lie in $S(\mathbb{T}) = \mathfrak{B}_2(-2)$. The system is exponentially stable.

The following example shows that even if the matrix A is not diagonalizable, but $\sigma(A) \subset S(\mathbb{T})$, then (1) may be exponentially stable.

Example 3.19: Let us consider the linear dynamic system of the following form

$$x^{\Delta} = \begin{pmatrix} -1 & 1 \\ -1 & -3 \end{pmatrix} x, \qquad (7)$$

defined on some time scale $\mathbb{T}$. The eigenvalues of the coefficient matrix A in (6) are $\lambda_1 = \lambda_2 = -2$ and A is not diagonalizable. Assume that $1 - 2\mu(t) \neq 0$ for all $t \in \mathbb{T}$. Using the Putzer Algorithm (see Theorem 5.35 in [2]) we obtain

$$x(t) = e_{-2}(t,t_0)\left[C_1\begin{pmatrix} 1+\int_{t_0}^{t}\frac{\Delta\tau}{1-2\mu(\tau)} \\ -\int_{t_0}^{t}\frac{\Delta\tau}{1-2\mu(\tau)} \end{pmatrix} + C_2\begin{pmatrix} \int_{t_0}^{t}\frac{\Delta\tau}{1-2\mu(\tau)} \\ 1-\int_{t_0}^{t}\frac{\Delta\tau}{1-2\mu(\tau)} \end{pmatrix}\right],$$

for $t \geqslant t_0$.

Let us take $\mathbb{T} = \mathbb{R}$ and $t_0 = 0$. Then $\sigma(A) \subset \{\lambda : Re\lambda < 0\} = S(\mathbb{R})$ and $e_{-2}(t,0) = e^{-2t}$, $\int_0^t \frac{\Delta\tau}{1-2\mu(\tau)} = \int_0^t dt = t$. Hence $x(t) = e^{-2t}\left[C_1\begin{pmatrix} 1+t \\ -t \end{pmatrix} + C_2\begin{pmatrix} t \\ 1-t \end{pmatrix}\right]$ and

$$\begin{aligned} \|x(t)\| &\leqslant \left(|C_1|\sqrt{(1+t)^2+t^2} + |C_2|\sqrt{(1-t)^2+t^2}\right)e^{-2t} \\ &\leqslant \sqrt{2}(|C_1|+|C_2|)e^{-t}, \quad \text{for } t \geqslant 0. \end{aligned}$$

Hence the solution of (7) is exponentially stable on $\mathbb{R}$.

Let $\mathbb{T} = \mathbb{P}_{1,1} := \bigcup_{k=0}^{\infty}[2k, 2k+1]$ and $t_0 = 0$. Then $e_{-1}(t,0) = (-1)^k e^{-2t}$ and $\int_0^t \frac{\Delta\tau}{1-2\mu(\tau)} = t - 2k$, for $2k \leqslant t \leqslant 2k+1$. Hence $x(t) = (-1)^k e^{-2t}\left[C_1\begin{pmatrix} 1+t-2k \\ -t+2k \end{pmatrix} + C_2\begin{pmatrix} t-2k \\ 1-t+2k \end{pmatrix}\right]$ and

$$\begin{aligned} \|x(t)\| &\leqslant e^{-2t}\left[|C_1|\sqrt{(1+t-2k)^2+(-t+2k)^2} + \right. \\ &\left. + |C_2|\sqrt{(t-2k)^2+(1-t+2k)^2}\right] \leqslant \\ &\leqslant e^{-2t}\left[\sqrt{5}|C_1| + \sqrt{2}|C_2|\right], \end{aligned}$$

for $t \geqslant 0$ and $2k \leqslant t \leqslant 2k+1$, $k \in \mathbb{N}_0$. Therefore the solution of (7) is exponentially stable for $\mathbb{T} = \mathbb{P}_{1,1}$.

IV. Stabilization and Controllability

Assume that $\sup\mathbb{T} = \infty$.

Let us consider the linear control system defined on the time scales $\mathbb{T}$

$$x^{\Delta}(t) = Ax(t) + Bu(t), \qquad (8)$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$. We assume that the control function u is piecewise rd-continuous.

Then

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))Bu(\tau)\Delta\tau$$

is the solution of (8) with the initial condition $x(t_0) = x_0$ for $t \geqslant t_0$.

The definitions of stabilizability and controllability for systems defined on a time scale are standard.

Definition 4.1: System (8) is *stabilizable* if there is a feedback $u(t) = Kx(t)$, for $K \in \mathbb{R}^{m\times n}$, such that the closed loop system $x^{\Delta}(t) = (A+BK)x(t)$ is exponentially stable.

Definition 4.2: System (8) is *controllable* if for any two states $x_1, x_2 \in \mathbb{R}^n$ there exist $t_1, t_2 \in \mathbb{T}$, $t_1 < t_2$, and a piecewise rd-continuous control $u(t)$, $t \in [t_1, t_2] \cap \mathbb{T}$ such that for $x_1 = x(t_1)$ one has $x(t_2) = x_2$.

The Kalman condition of controllability can be extended to linear control systems defined on time scales. The following theorem holds

Theorem 4.3: [10] System (8) is controllable if and only if the following condition holds:

$$\operatorname{rank}\left[B, AB, A^2B, \ldots, A^{n-1}B\right] = n.$$

As controllability of linear control systems does not depend on the time scale, we have another standard characterization.

Theorem 4.4: [11] The following conditions are equivalent:

(i) system (8) is controllable
(ii) $\operatorname{rank}\left[B, AB, A^2B, \ldots, A^{n-1}B\right] = n$
(iii) $\operatorname{rank}[\lambda I - A, B] = n$, for every $\lambda \in \mathbb{C}$
(iv) for every $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{R}$ there exists a matrix $K \in \mathbb{R}^{m\times n}$ such that

$$\chi_{A+BK}(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \ldots + \alpha_1\lambda + \alpha_0,$$

where χ_{A+BK} is the characteristic polynomial of the matrix $A+BK$.

From Theorem 3.17 we see that if only the matrix A is diagonalizable, then its spectrum decides about exponential stability of (1).

Theorem 4.5: Assume that $\mu(t)$ is bounded. If system (8) is controllable, then it is stabilizable.

Proof: Assume that system (8) is controllable. Then $\operatorname{rank}\left[B, AB, \ldots, A^{n-1}B\right] = n$. Because the graininess function is bounded, then $S_{\mathbb{C}}(\mathbb{T}) \neq \emptyset$ and so $S(\mathbb{T})$ is infinite. Hence, by Theorem 4.4, we can choose a matrix K such that $\sigma(A+BK) \subset S(\mathbb{T})$ and all the eigenvalues of $A+BK$ are distinct. This means that $A+BK$ is diagonalizable. From Theorem 3.17 the system $x^\Delta(t) = (A+BK)x(t)$ is exponentially stable, so (8) is stabilizable. ∎

Example 4.6: Let $\mathbb{T} = 2^{\mathbb{N}}$, $n = 1$, $m = 1$, $A = 0$ and $B = 1$. Then (8) is controllable but it is not stabilizable.

Similarly as in the continuous-time case one can show the following theorem.

Theorem 4.7: Assume that $\mu(t)$ is bounded. System (8) is stabilizable if and only if the following implication holds

$$\operatorname{rank}[\lambda I - A, B] < n \Rightarrow \lambda \in S(\mathbb{T}).$$

V. Detectability and observability

The dual concept to controllability is observability and to stabilization is detectability.

Let $\sup \mathbb{T} = \infty$ as before. Let us consider the linear control system with output defined on the time scale $\mathbb{T}$

$$\begin{aligned} x^\Delta(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \tag{9}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$.

If the state of (9) is unknown then we can construct an observer whose task is an estimation of the state.

Definition 5.1: The following system

$$\begin{aligned} z^\Delta(t) &= Fz(t) + Ly(t) + Gu(t), \\ \hat{x}(t) &= Ny(t) + Mz(t), \end{aligned} \tag{10}$$

where $z(t) \in \mathbb{R}^q$ is the state of (10), $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are its inputs and are taken from (9), $F \in \mathbb{R}^{q\times q}$, $G \in \mathbb{R}^{q\times m}$, $L \in \mathbb{R}^{q\times p}$, $N \in \mathbb{R}^{n\times p}$ and $M \in \mathbb{R}^{n\times q}$, is called the *q-dimensional observer* of (9) if $\lim_{t\to+\infty}[\hat{x}(t) - x(t)] = 0$.

Definition 5.2: The system (9) is *detectable* if there is a matrix $L \in \mathbb{R}^{n\times p}$ such that the system $\varepsilon^\Delta(t) = (A+LC)\varepsilon(t)$ is exponentially stable.

Remark 5.3: If (9) has a n-dimensional (full order) observer of the following form $\hat{x}^\Delta(t) = (A+LC)\hat{x}(t) - Ly(t) + Bu(t)$, then the error $\varepsilon(t) := \hat{x}(t) - x(t)$ satisfies the equation in Definition 5.2.

As we see from Remark 5.3, detectability of linear control systems is connected with existence of full order observers.

Definition 5.4: Two states $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* (with respect to (9)) if for every $t_0 \in \mathbb{T}$, for every control u defined on $[t_0, \infty) \cap \mathbb{T}$ and for every time $t \in [t_0, \infty) \cap \mathbb{T}$, the value of the output $y(t)$ corresponding to u is the same for both initial conditions $x(t_0) = x_1$ and $x(t_0) = x_2$. System (9) is *observable* if any two indistinguishable states are equal.

Observability of systems on an arbitrary time scale is characterized by the standard Kalman condition.

Theorem 5.5: [10] System (9) is observable if and only if the following condition holds:

$$\operatorname{rank}\left[C^{\mathrm{T}}, (CA)^{\mathrm{T}}, \left(CA^2\right)^{\mathrm{T}}, \ldots, \left(CA^{n-1}\right)^{\mathrm{T}}\right] = n.$$

Theorem 5.6: Let the graininess function μ be bounded. If system (9) is observable, then it is detectable and the following equation

$$\hat{x}^\Delta(t) = (A+LC)\hat{x}(t) - Ly(t) + Bu(t) \tag{11}$$

defines an n-dimensional observer.

Proof: Assume that system (9) is observable. Then $\operatorname{rank}\left[C^{\mathrm{T}}, (CA)^{\mathrm{T}}, (CA^2)^{\mathrm{T}}, \ldots, (CA^{n-1})^{\mathrm{T}}\right] = n$. Because the graininess function is bounded, then $S(\mathbb{T}) \neq \emptyset$ and infinite. Hence, by Theorem 4.4, we can choose a matrix L such that $A+LC$ is diagonalizable and $\sigma(A+LC) \subset S(\mathbb{T})$. From Theorem 3.17 the system $\varepsilon^\Delta(t) = (A+LC)\varepsilon(t)$ is exponentially stable, where $\varepsilon(t) = \hat{x}(t) - x(t)$. Hence system (9) is detectable and (11) is its observer. ∎

Similarly as in the continuous-time case one can show the following theorem.

Theorem 5.7: Assume that the graininess function μ is bounded. System (9) is detectable if and only if the following condition holds

$$\operatorname{rank}\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} < n \Rightarrow \lambda \in S(\mathbb{T}).$$

Equation (10) defines the full order observer, because both $x(t)$ and $\hat{x}(t)$ belong to the same space, namely $\mathbb{R}^n$. Since from the output $y(t)$ of (9) we can get directly some of the components of the unknown state, using this information we need only find an estimation of the remaining coordinates of the state. If $\operatorname{rank} C = p > 0$, then we are able to compute p coordinates of the state vector from the output. Then the task of the observer is the reconstruction of $n-p$ remaining coordinates. Such observer is of the order $n-p$ and it is called a reduced order observer.

Assume that system (9) is observable, the graininess function is bounded and $\mathrm{rank} C = p > 0$. Let us consider the following system

$$\bar{x}^{\Delta}(t) = F\bar{x}(t) + Gu(t) + Hy(t), \tag{12}$$

where $t \in \mathbb{T}$, $\bar{x}(t) \in \mathbb{R}^{n-p}$ is the state vector, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are inputs of (12) and are taken from system (9), and $F \in \mathbb{R}^{(n-p)\times(n-p)}$, $G \in \mathbb{R}^{(n-p)\times m}$, $H \in \mathbb{R}^{(n-p)\times p}$.

Let $\varepsilon(t) := \bar{x}(t) - Tx(t)$ be the error vector, where $T \in \mathbb{R}^{(n-p)\times n}$ and we choose the matrix T so as $\det\begin{pmatrix} T \\ C \end{pmatrix} \neq 0$. Then

$$\begin{aligned} \varepsilon^{\Delta}(t) &= \bar{x}^{\Delta}(t) - Tx^{\Delta}(t) = \\ &= (FT - (TA - HC))x(t) + F\varepsilon(t) + (G - TB)u(t). \end{aligned}$$

If $TA = FT + HC$ and $G = TB$, then we obtain $\varepsilon^{\Delta}(t) = F\varepsilon(t)$. Therefore $\varepsilon(t) = e_F(t,t_0)\varepsilon(t_0)$, where $t_0 \in \mathbb{T}$, $t \geqslant t_0$ and $\varepsilon(t_0) = \bar{x}(t_0) - Tx(t_0)$.

The estimation $\hat{x}$ of the state vector x satisfies the following condition

$$\begin{pmatrix} T \\ C \end{pmatrix} \cdot \hat{x}(t) = \begin{pmatrix} \bar{x}(t) \\ y(t) \end{pmatrix},$$

for all $t \geqslant t_0$ and $t \in \mathbb{T}$. If $\begin{pmatrix} M & N \end{pmatrix} := \begin{pmatrix} T \\ C \end{pmatrix}^{-1}$, then

$$\hat{x}(t) = \begin{pmatrix} M & N \end{pmatrix} \cdot \begin{pmatrix} \bar{x}(t) \\ y(t) \end{pmatrix},$$

for all $t \geqslant t_0$ and $t \in \mathbb{T}$.

Hence we obtain $\hat{x}(t) = M\bar{x}(t) + Ny(t) = M(Tx(t) + \varepsilon(t)) + NCx(t) = Me_F(t,t_0)\varepsilon(0) + (MT + NC)x(t)$. Because $MT + NC = I$, so $\hat{x}(t) - x(t) = Me_F(t,t_0)\varepsilon(t)$ for $t \geqslant t_0$ and $t \in \mathbb{T}$.

If the matrix F is chosen in such a way that the system $\varepsilon^{\Delta}(t) = F\varepsilon(t)$ is exponentially stable, then $\lim_{t\to\infty} \varepsilon(t) = 0$ and $\lim_{t\to\infty} [\hat{x}(t) - x(t)] = 0$.

Note that $TA = \begin{pmatrix} F & H \end{pmatrix} \cdot \begin{pmatrix} T \\ C \end{pmatrix}$ and $\begin{pmatrix} F & H \end{pmatrix} = TA \cdot \begin{pmatrix} M & N \end{pmatrix}$, so $F = TAM$ and $H = TAN$. The matrix T can be chosen in such a way that the system $\varepsilon^{\Delta}(t) = TAM\varepsilon(t)$ is exponentially stable (see [12]). Then the following system

$$\begin{aligned} \bar{x}^{\Delta}(t) &= TAM\bar{x}(t) + TBu(t) + TANy(t), \\ \hat{x}(t) &= M\bar{x}(t) + Ny(t) \end{aligned} \tag{13}$$

is the reduced order observer ($(n-p)$–dimensional observer) of the linear system (9).

VI. CONCLUSIONS AND FUTURE WORKS

A. Conclusions

The exponential stability of a linear (control) system is connected with the set of exponential stability which depends on a time scale. It may happen that this set is empty and no system is then exponentially stable. Boundedness of the graininess function guarantees that the set of exponential stability is not empty. However there are some time scales with unbounded graininess function for which the set of exponential stability is nonempty.

If a control system on a time scale is not exponentially stable but is controllable, it may be stabilized via the standard feedback. This is however possible only if the set of exponential stability is nonempty. As controllability is characterized by the same condition for all time scales, most of the results known for the continuous-time case can be extended to arbitrary time scales.

The same happens to existence of observers, detectability and observability. Once the set of exponential stability is nonempty, observers can be constructed in a similar fashion as for continuous-time or discrete-time systems.

B. Future Works

Understanding stability for systems on time scales is far from being complete. Further study of the sets of exponential stability seems to be necessary. Stability, stabilization and observers of nonlinear systems on time scales have not been studied yet. As many results are known for continuous-time and discrete-time systems, one may be tempted to try to unify these, sometimes distant, theories.

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