IRREDUCIBILITY CONDITIONS FOR NONLINEAR INPUT-OUTPUT EQUATIONS ON HOMOGENEOUS TIME SCALES

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Abstract: The purpose of this paper is to present a necessary and sufficient condition for irreducibility of nonlinear input-output delta differential equations. The condition is presented in terms of the common left factor of two differential polynomials describing the behaviour of the system defined on a homogenous time scale. *Copyright © 2007 IFAC*

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1. INTRODUCTION

In the realization problem of an higher order input-output (i/o) difference or differential equation one is looking for the state equations that would generate a given i/o equation. In (Kotta *et al.*, 2001) and (Liu and Moog, 1994; Moog *et al.*, 2002) it was shown for discrete- and continuous-time systems, respectively, that to obtain a minimal (i.e both observable and accessible) realization of an i/o equation, the i/o equation has to be irreducible. So, the first task in solving the realization problem is to reduce the i/o equation when necessary. The irreducibility may be checked and the reduced system can be found in many different ways, see (Kotta *et al.*, 2001; Kotta, 2000; Kotta *et al.*, 2004) for the discrete-time case and (Moog *et al.*, 2002; Zheng *et al.*, 2001) for continuous-time case. One possibility is to associate with the control system two polynomials, defined over the difference (or differential - in the continuous time case) field of meromorphic functions, pretty much in the similar manner it has be done in the linear case. Then, in practical terms, checking irreducibility requires to find the left common factor of these two polynomials. Both in discrete- and continuous-time case the criteria for checking irreducibility are based on the similar ideas. The main difference is that the multiplication rules between the shift (or differentiation in the continuous-time case) operator and an element of difference (resp. differential) field are different and yield different non-commutative ring of polynomials. Therefore, it seems natural to try to unify the

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results for discrete- and continuous-time cases into one result from which both would follow. However, in the discrete-time case our formalism yields a description based on the difference operator in opposition to the shift operator which was used in (Kotta, 2000).

The calculus on time scales, originated by S. Hilger (Hilger, 1988), seems to be a perfect language for unification of continuous- and discrete-time cases. A time scale is a model of time. It is an arbitrary closed subset of the real line. Besides of standard cases of the whole line (continuous-time case) and the set of integers (discrete-time case), there are many examples of time models that may be partly continuous and partly discrete. One of the main concepts in the time scale analysis is the delta derivative, which is a generalization of the classical (time) derivative in the continuous time and the finite forward difference in the discrete time. It can be also stressed that the differential calculus on time scales can be implemented in `Mathematica` (Yantir, 2003).

The purpose of this paper is to present a necessary and sufficient condition for irreducibility of nonlinear i/o delta differential equation on homogeneous time scale that accommodates both the discrete- and continuous-time cases as the special cases. We will show that one can associate with such system two polynomials over the σ-differential field, that belong to a non-commutative skew polynomial ring.

2. CALCULUS ON TIME SCALE

For a general introduction to the calculus on time scales, see (Bohner and Peterson, 2001). Here we give only those notions and facts that we need in our paper.

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, but also $\mathbb{T} = q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$ is a time scale. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ for $t \in \mathbb{T}$, while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. Additionally, we set $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$ and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, and if $t > \min T$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-scattered and left-scattered at the same time are called *isolated*. Finally, the graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$, for all $t \in \mathbb{T}$.

If $\mu \equiv$ const then a time scale $\mathbb{T}$ is called *homogeneous*.

Let $\mathbb{T}^\kappa$ denote truncated set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point.

Definition 1. Let $f : \mathbb{T} \to \mathbb{R}$ be a function and $t \in \mathbb{T}^\kappa$. Then the *delta derivative* of f at the point t is defined to be the number $f^\Delta(t)$ (provided it exists) with the property that for each $\epsilon > 0$ there exists a neighbourhood $\mathcal{U}$ of t in $\mathbb{T}$ such that

$$| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] | \leqslant \epsilon \, | \sigma(t) - s |,$$

for all $s \in \mathcal{U}$. Moreover, we say that f is *delta differentiable* (on $\mathbb{T}^\kappa$) provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

Let $f : \mathbb{T} \to \mathbb{R}$. Then we define $f^\sigma := f \circ \sigma$.

Proposition 2. Let $f : \mathbb{T} \to \mathbb{R}$, $g : \mathbb{T} \to \mathbb{R}$ be two delta differentiable functions defined on $\mathbb{T}$ and let $t \in \mathbb{T}$. Then

(*i*) f is continuous at t.
(*ii*) $f^\sigma = f + \mu f^\Delta$.
(*iii*) $[\alpha f + \beta g]^\Delta = \alpha f^\Delta + \beta g^\Delta$, for any constants α and β.
(*iv*) $(fg)^\Delta = f^\sigma g^\Delta + f^\Delta g$.
(*v*) if $gg^\sigma \neq 0$, then $\left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - f g^\Delta}{g g^\sigma}$.

Theorem 3. (Chain Rule). Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \to \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \to \mathbb{R}$ is delta differentiable and

$$(f \circ g)^\Delta(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t) g^\Delta(t)) \mathrm{d}h \right\} g^\Delta(t)$$

For a function $f : \mathbb{T} \to \mathbb{R}$ we can talk about second delta derivative $f^{[2]} := f^{\Delta\Delta}$ provided that f^Δ is delta differentiable on $\mathbb{T}^{\kappa^2} := (\mathbb{T}^\kappa)^\kappa$ with delta derivative $f^{[2]} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly we define higher order delta derivatives $f^{[n]} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$.

Let us denote $f^{\Delta\sigma} := \left(f^\Delta\right)^\sigma$ and $f^{\sigma\Delta} := (f^\sigma)^\Delta$. Note that if f, f^Δ and μ are delta differentiable, then

$$f^{\sigma\Delta} = \left(1 + \mu^\Delta\right) f^{\Delta\sigma}. \qquad (1)$$

Remark 4. Let f and f^Δ be delta differentiable functions. Then for a homogeneous time scale $\mathbb{T}$ we have $f^{\Delta\sigma} = f^{\sigma\Delta}$.

Let $\sigma^n := \underbrace{\sigma \circ \cdots \circ \sigma}_{n-\text{times}}$ and $f^{\sigma^n} := f \circ \sigma^n$. By the induction principle one can prove that if f is a delta differentiable function defined on a homogeneous time scale $\mathbb{T}$, then

$$f^{\sigma^n} = \sum_{k=0}^{n} \binom{n}{k} \mu^k f^{[k]}.$$

3. EXTENDED SYSTEM

From now on we assume that the time scale $\mathbb{T}$ is homogenous. For $f : \mathbb{T} \to \mathbb{R}$ and $i \leqslant k$ let $f^{[i..k]} := \left(f^{[i]}, \ldots, f^{[k]}\right)$.

Let $y : \mathbb{T} \to \mathbb{R}$ and $u : \mathbb{R} \to \mathbb{T}$ be delta differentiable functions.

Consider a single-input single-output dynamical system described by a higher order input-output (i/o) delta-differential equation on time scale $\mathbb{T}$:

$$\varphi\left(y^{[0..n]}, u^{[0..s]}\right) = 0, \qquad (2)$$

where u is a scalar input variable, $y \in \mathcal{Y} \subset \mathbb{R}$ is a scalar output variable, φ is a real analytic function defined on $\mathcal{Y}^{n+1} \times \mathbb{R}^{s+1}$, n, s are nonnegative integers such that $s < n$.

Assume that φ cannot be decomposed into a product of simpler functions.

Additionally, we assume that

$$\left(\frac{\partial \varphi}{\partial y^{[n]}}\right)(0, \ldots, 0) \neq 0. \qquad (3)$$

It means that equation (2) can be solved uniquely in the neighbourhood of $(0, \ldots, 0)$ for $y^{[n]}$:

$$y^{[n]} = \phi\left(y^{[0..n-1]}, u^{[0..s]}\right). \qquad (4)$$

One can associate with system (4) an extended state-space system Σ_e with input $v = u^{[s+1]}$ and state $z = \left[y^{[0..n-1]}, u^{[0..s]}\right]^{\mathrm{T}}$ defined as

$$\begin{aligned} z_1^{\Delta} &= z_2 \\ z_2^{\Delta} &= z_3 \\ &\vdots \\ z_{n-1}^{\Delta} &= z_n \\ z_n^{\Delta} &= \phi(z) \\ z_{n+1}^{\Delta} &= z_{n+2} \\ &\vdots \\ z_{n+s}^{\Delta} &= z_{n+s+1} \\ z_{n+s+1}^{\Delta} &= v. \end{aligned} \qquad (5)$$

Denote $[z_2, z_3, \ldots, z_n, \phi(z), z_{n+2}, \ldots, z_{n+s+1}, v]^{\mathrm{T}}$ by $f_e(z, v)$. Then equations (5) can be written in a vector form as $z^{\Delta} = f_e(z, v)$. We assume that the map $z \mapsto \tilde{f}(z, v) = z + \mu f_e(z, v)$ satisfies generically the following condition

$$\operatorname{rank} \frac{\partial \tilde{f}(z, v)}{\partial (z_1, \ldots, z_{n+s+1}, v)} = n + s + 1. \qquad (6)$$

Let $\mathcal{R}$ denote the ring of analytic functions in a finite number of the variables $\{y^{[0..n-1]}, u^{[k]} : k \geqslant 0\} = \{z_1, \ldots, z_{n+s+1}, v^{[l]} : l \geqslant 0\}$. The operators $\sigma : \mathcal{R} \to \mathcal{R}$ and $\Delta : \mathcal{R} \to \mathcal{R}$ are defined as follows

$$\sigma(F)\left(z, v^{[0..l+1]}\right) := F\left(z^{\sigma}, \left(v^{[0..l]}\right)^{\sigma}\right), \qquad (7)$$

where $z = (z_1, \ldots, z_{n+s+1})$ and $z^{\sigma} = z + \mu f_e(z, v)$, $\left(v^{[0..l]}\right)^{\sigma} = u^{[0..l]} + \mu u^{[1..l+1]}$, $l \geqslant 0$ and

$$\Delta(F)\left(z, v^{[0..l+1]}\right) := \int_0^1 \{\operatorname{grad} F(z + h\mu f_e(z, v), v^{[0..l]} + h\mu v^{[1..l+1]}) \cdot \begin{bmatrix} f_e(z, v) \\ \left(v^{[1..l+1]}\right)^{\mathrm{T}} \end{bmatrix} \} dh. \qquad (8)$$

We will use $\sigma(F)$ and F^{σ} to denote the action of σ on F. Similarly, both $\Delta(F)$ and F^{Δ} will be used interchangeably. Note that (6) implies that σ is a monomorphism of $\mathcal{R}$. Let $\mathcal{K}$ be a quotient field of the ring $\mathcal{R}$. Elements of $\mathcal{K}$ are meromorphic functions depending on a finite number of variables from $\{z, v^{[l]}, l \geqslant 0\}$. The operators σ and Δ can be extended to $\mathcal{K}$ using the same formulas (7) and (8).

The operator Δ satisfies a generalization of Leibniz rule:

$$(FG)^{\Delta} = F^{\sigma} G^{\Delta} + F^{\Delta} G, \qquad (9)$$

for $F, G \in \mathcal{K}$. In noncommutative algebra the derivation satisfying rule (9) is called a *"σ-derivation"* (for example see (Cohn, 1965)). Therefore $\mathcal{K}$ is a field equipped with a σ-derivation Δ such that σ is a monomorphism of $\mathcal{K}$. The field $\mathcal{K}$ with σ-derivation Δ is a *σ-differential field.* Since σ is one-to-one, one can show that there is a σ-differential overfield $\mathcal{K}^*$, called the *inversive closure* of $\mathcal{K}$, such that σ can be extended to $\mathcal{K}^*$ and this extension is an automorphism of $\mathcal{K}^*$ (see (Cohn, 1965)). Therefore we assume that the inversive closure of σ-differential field $\mathcal{K}$ is given and we will use the same symbol to denote the σ-differential field and its inversive closure.

4. RINGS OF σ-DIFFERENTIAL POLYNOMIALS

In this section we consider an arbitrary σ-differential field $\mathcal{L}$ with the σ-derivation Δ. We shall assume that σ is an automorphism of $\mathcal{L}$. A *left* differential polynomial is an element which can be uniquely written in the form

$$a(\partial) = \sum_{i=0}^{n} a_i \partial^{n-i}, \quad a_i \in \mathcal{L} \qquad (10)$$

where ∂ is a formal variable and $a(\partial) \neq 0$ if and only if at least one of the functions a_i, $i = 0, \ldots, n$ is nonzero. If $a_0 \not\equiv 0$, then the positive integer n is called the *degree* of the left polynomial $a(\partial)$ and denoted by $\mathrm{d}^0(a)$. In addition, we set $\mathrm{d}^0(0) = -\infty$.

For $a \in \mathcal{L}$ let us define multiplication

$$\partial \cdot a := a^{\sigma} \cdot \partial + a^{\Delta}. \tag{11}$$

This rule can be uniquely extended to multiplication on monomials by

$$(a\partial^n)(b\partial^m) = (a\partial^{n-1})(b^{\sigma}\partial^{m+1} + b^{\Delta}\partial^m).$$

The ring of differential polynomials will be denoted by $\mathcal{L}[\partial; \sigma, \Delta]$. Since σ is an automorphism, the ring of the left differential polynomials is an Ore ring (McConnell and Robson, 1987).

Let $\sigma^n := \underbrace{\sigma \circ \sigma \circ \ldots \circ \sigma}_{n-\text{times}}$ and denote $\sigma^n(a)$ by a^{σ^n}, for $a \in \mathcal{L}$. Using the induction principle the following lemma can be proved:

Lemma 5. Let $a \in \mathcal{L}$. Then $\partial^n \cdot a \in \mathcal{L}[\partial; \sigma, \Delta]$, $n \geqslant 1$, and $\partial^n \cdot a = \sum\limits_{k=0}^{n} \binom{n}{k} \left(a^{[n-k]}\right)^{\sigma^k} \partial^k$.

A ring D is called an *integral domain*, or a *domain*, if it does not contain any zero divisors. This means that if a and b are two elements of D such that $ab = 0$, then $a = 0$ or $b = 0$.

Proposition 6. (McConnell and Robson, 1987)

(*i*) The ring $\mathcal{L}[\partial; \sigma, \Delta]$ is an integral domain.
(*ii*) If a and b are nonzero differential polynomials, then $\mathrm{d}^0(ab) = \mathrm{d}^0(a) + \mathrm{d}^0(b)$.

In next sections we will use left differentials polynomials over the field $\mathcal{K}$, that was introduced in Section 3.

5. A POLYNOMIAL DESCRIPTION OF THE NONLINEAR INPUT-OUTPUT DELTA-DIFFERENTIAL EQUATION

Over the σ-differential field $\mathcal{K}$ one can define the following vector space

$$\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y, \mathrm{d}y^{\Delta}, \ldots, \mathrm{d}y^{[n-1]}, \mathrm{d}u^{[k]} :\ k \geqslant 0\}.$$

The elements of $\mathcal{E}$ will be called one-forms.

For $F \in \mathcal{K}$ we define $\mathrm{d} : \mathcal{K} \to \mathcal{E}$ as follows

$$\mathrm{d}F := \sum_{i=0}^{n-1} \frac{\partial F}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \sum_{l=0}^{k} \frac{\partial F}{\partial u^{[l]}} \mathrm{d}u^{[l]}. \tag{12}$$

$\mathrm{d}F$ is said to be the *total differential* (or simply the *differential*) of the function F and it is a one-form.

If $\omega = \sum\limits_i \alpha_i \mathrm{d}\zeta_i$ is a one-form, where $\alpha_i \in \mathcal{K}$ and $\zeta_i \in \{y, \ldots, y^{[n-1]}, u^{[k]} :\ k \geqslant 0\}$, then the operators $\Delta : \mathcal{K} \to \mathcal{K}$ and $\sigma : \mathcal{K} \to \mathcal{K}$ induce operators $\Delta : \mathcal{E} \to \mathcal{E}$ and $\sigma : \mathcal{E} \to \mathcal{E}$ by

$$\Delta(\omega) \quad := \quad \sum_i \left\{\alpha_i^{\Delta} \mathrm{d}\zeta_i + \alpha_i^{\sigma} \mathrm{d}\left[\zeta_i^{\Delta}\right]\right\}, \tag{13}$$

$$\sigma(\omega) := \sum_i \alpha_i^{\sigma} \mathrm{d}\left[\zeta_i^{\sigma}\right]. \tag{14}$$

One can show that $\Delta(\mathrm{d}F) = \mathrm{d}[F^{\Delta}]$, $\sigma(\mathrm{d}F) = \mathrm{d}[F^{\sigma}]$ and $\Delta\sigma = \sigma\Delta$.

Let us define $\partial^k \mathrm{d}y := \mathrm{d}y^{[k]}$ and $\partial^l \mathrm{d}u := \mathrm{d}u^{[l]}$, $k, l \geqslant 0$ in the vector space $\mathcal{E}$. Since every one-form $\omega \in \mathcal{E}$ has the following form $\omega = \sum\limits_{i=0}^{n-1} a_i \mathrm{d}y^{[i]} + \sum\limits_{j=0}^{k} b_j \mathrm{d}u^{[j]}$, where $a_i, b_j \in \mathcal{K}$, so ω can be expressed in terms of the left differential polynomials in the following way

$$\omega = \left(\sum_{i=0}^{n-1} a_i \partial^i\right) \mathrm{d}y + \left(\sum_{j=0}^{k} b_j \partial^j\right) \mathrm{d}u.$$

A left differential polynomial can be considered as an operator on vectors $\mathrm{d}y$ and $\mathrm{d}u$ from $\mathcal{E}$:

$$\left(\sum_{i=0}^{k} a_i \partial^i\right) (\alpha \mathrm{d}\vartheta) := \sum_{i=0}^{k} a_i \left(\partial^i \cdot \alpha\right) \mathrm{d}\vartheta,$$

with $a_i, \alpha \in \mathcal{K}$ and $\mathrm{d}\vartheta \in \{\mathrm{d}y, \mathrm{d}u\}$. It is easy to notice that $\partial(\omega) = \Delta(\omega)$, for $\omega \in \mathcal{E}$. Additionally, by Lemma 5 one can show the following proposition:

Proposition 7. Let $F \in \mathcal{K}$ and $n \in \mathbb{N}$. Then

$$\partial^n (\mathrm{d}F) = \mathrm{d}F^{[n]}. \tag{15}$$

Instead of working with equation (4) describing the control system, we can work with its differential

$$\mathrm{d}y^{[n]} - \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial y^{[i]}} \mathrm{d}y^{[i]} - \sum_{j=0}^{s} \frac{\partial \phi}{\partial u^{[j]}} \mathrm{d}u^{[j]} = 0. \tag{16}$$

Since $\mathrm{d}y^{[i]} = \partial^i \mathrm{d}y$ and $\mathrm{d}u^{[j]} = \partial^j \mathrm{d}u$, (16) can be rewritten as

$$p(\partial)\mathrm{d}y = q(\partial)\mathrm{d}u, \tag{17}$$

where

$$p(\partial) = \partial^n - \sum_{i=1}^{n-1} \frac{\partial \phi}{\partial y^{[i]}} \partial^i,$$

$$q(\partial) = \sum_{j=0}^{s} \frac{\partial \phi}{\partial u^{[j]}} \partial^j$$

and $\frac{\partial \phi}{\partial y^{[i]}} \in \mathcal{K}$, $\frac{\partial \phi}{\partial u^{[j]}} \in \mathcal{K}$. Equation (17) describes the dynamical system behaviour in terms of two differential polynomials $p(\partial)$, $q(\partial)$ in delta-differential operator ∂ over the σ-differential field $\mathcal{K}$.

Since $\mathcal{K}[\partial;\sigma,\Delta]$ is an Ore ring, so one can construct the division ring of fractions. If $p(\partial) = p_1(\partial)p_2(\partial)$, then $p_1(\partial)$ is called a *left divisor* of $p(\partial)$ and $p(\partial)$ is called *left divisible* by $p_1(\partial)$. If for $p_1(\partial), p_2(\partial) \in \mathcal{K}[\partial;\sigma,\Delta]$, $p_c(\partial)$ is a left divisor of $p_1(\partial) - p_2(\partial)$, then $p_c(\partial)$ is called a *common left factor* of $p_1(\partial)$ and $p_2(\partial)$. If the degree of $p_c(\partial)$ is the greatest of all common left factors of $p_1(\partial) - p_2(\partial)$, then $p_c(\partial)$ is called the *greatest common left factor.*

To find the greatest common left factor one can use the *left Euclidean division algorithm*, (Bronstein and Petkovšek, 1996). To perform the left Euclidean division algorithm it is sufficient that σ be an automorphism. For given two polynomials $p_1(\partial)$ and $p_2(\partial)$ with $\mathrm{d}^0 p_1(\partial) > \mathrm{d}^0 p_2(\partial)$ there exists a unique polynomial $\gamma_1(\partial)$ and a unique left remainder polynomial $p_3(\partial)$ such that

$$p_1(\partial) = p_2(\partial)\gamma_1(\partial) + p_3(\partial),\ \mathrm{d}^0 p_3(\partial) < \mathrm{d}^0 p_2(\partial).$$

Using (the left) Euclidean division algorithm, after k steps, we obtain

$$\begin{aligned} p_2(\partial) &= p_3(\partial)\gamma_2(\partial) + p_4(\partial) \\ &\vdots \\ p_{k-2}(\partial) &= p_{k-1}(\partial)\gamma_{k-2}(\partial) + p_k(\partial) \\ p_{k-1}(\partial) &= p_k(\partial)\gamma_{k-1}(\partial). \end{aligned}$$

Hence the greatest common left divisor of $p_1(\partial)$ and $p_2(\partial)$ is $p_k(\partial)$. The greatest common left divisor is only unique up to meromorphic functions from $\mathcal{K}$, but it can be made unique by requiring it to be monic. Thus for two polynomials $\tilde{p}_1(\partial)$ and $\tilde{p}_2(\partial)$ we have

$$\begin{aligned} p_1(\partial) &= p_k(\partial)\tilde{p}_1(\partial), \\ p_2(\partial) &= p_k(\partial)\tilde{p}_2(\partial). \end{aligned}$$

If $\mathrm{d}^0 p_k(\partial) = 0$, then the polynomials $p_1(\partial)$ and $p_2(\partial)$ have no common left divisor and are called *left coprime* (*relatively prime*).

6. IRREDUCIBILITY OF THE I/O DELTA-DIFFERENTIAL EQUATION

Definition 8. A function φ_r in $\mathcal{K}$, such that $\varphi_r(0,\ldots,0) = 0$, is said to be an *autonomous element* for control system (2) if there exist an integer $\nu \geqslant 1$ and a non-zero analytic function F so that

$$F\left(\varphi_r, \varphi_r^{\Delta}, \ldots, \varphi_r^{[\nu]}\right) = 0. \tag{18}$$

Definition 9. Control system (2) is said to be *irreducible (forward accessible)* if there does not exist any non-zero autonomous element in $\mathcal{K}$. Otherwise system (2) is called *reducible.*

If system (2) is not irreducible, then there exist an autonomous element $\varphi_r = \varphi_r\left(y^{[0..r]}, u^{[0..l]}\right)$ and a non-zero analytic function F such that (2) can be expressed as

$$kF\left(\varphi_r^{[0..\nu]}\right) = 0,$$

where $\nu \geqslant 1$ and $k \neq 0$ is an element of $\mathcal{K}$.

Theorem 10. Control system (4) is reducible in the sense of Definition 9 if and only if polynomials $p(\partial)$ and $q(\partial)$ in (17), associated to system (4) have common left divisors.

PROOF. "$\Leftarrow$" Suppose that nonlinear system (4) is reducible. Then there exist functions φ_r and F such that (18) holds. Let $\tilde{\varphi} := F\left(\varphi_r^{[0..\nu]}\right)$. Then $\tilde{\varphi} \in \mathcal{K}$ and

$$\mathrm{d}\varphi_r = \sum_{i=0}^{r} \frac{\partial \varphi_r}{\partial y^{[i]}} \mathrm{d}y^{[i]} + \sum_{j=0}^{l} \frac{\partial \varphi_r}{\partial u^{[j]}} \mathrm{d}u^{[j]} \tag{19}$$

and

$$\mathrm{d}\tilde{\varphi} = \sum_{k=0}^{\nu} \frac{\partial F}{\partial \varphi_r^{[k]}} \mathrm{d}\varphi_r^{[k]}. \tag{20}$$

By Proposition 7, $\partial^i \mathrm{d}y = \mathrm{d}y^{[i]}$, $\partial^j \mathrm{d}u = \mathrm{d}u^{[j]}$ and $\partial^k \mathrm{d}\varphi_r = \mathrm{d}\varphi_r^{[k]}$. Therefore (19) and (20) can be rewritten in terms of differential polynomials in operator ∂ over $\mathcal{K}$:

$$\mathrm{d}\varphi_r = \sum_{i=0}^{r} \frac{\partial \varphi_r}{\partial y^{[i]}} \partial^i \mathrm{d}y + \sum_{j=0}^{l} \frac{\partial \varphi_r}{\partial u^{[j]}} \partial^j \mathrm{d}u$$

and

$$\mathrm{d}\tilde{\varphi} = \sum_{k=0}^{\nu} \frac{\partial F}{\partial \varphi_r^{[k]}} \partial^k \mathrm{d}\varphi_r.$$

Let $\tilde{p}(\partial) := \sum_{i=0}^{r} \frac{\partial \varphi_r}{\partial y^{[i]}} \partial^i$, $\tilde{q}(\partial) := -\sum_{j=0}^{l} \frac{\partial \varphi_r}{\partial u^{[j]}} \partial^j$ and $\varsigma(\partial) := \sum_{k=0}^{\nu} \frac{\partial F}{\partial \varphi_r^{[k]}} \partial^k$. Then the equation $\mathrm{d}\tilde{\varphi} = 0$ can be rewritten as

$$\varsigma(\partial)\left[\tilde{p}(\partial)\mathrm{d}y - \tilde{q}(\partial)\mathrm{d}u\right] = 0.$$

Since $\{\mathrm{d}y, \ldots, \mathrm{d}y^{[n-1]}, \mathrm{d}u, \ldots, \mathrm{d}u^{[s]}\}$ is a set of independent vectors in the vector space $\mathcal{E}$, we obtain

$$p(\partial) = \varsigma(\partial)\tilde{p}(\partial),\ q(\partial) = \varsigma(\partial)\tilde{q}(\partial).$$

From the statement (ii) of Proposition 6 we have

$$\mathrm{d}^0 p = \mathrm{d}^0 \varsigma + \mathrm{d}^0 \tilde{p},\ \mathrm{d}^0 q = \mathrm{d}^0 \varsigma + \mathrm{d}^0 \tilde{q}.$$

Since $\mathrm{d}^0\varsigma = \nu \geqslant 1$, so $\mathrm{d}^0\varsigma > 0$. Hence the polynomials $p(\partial)$ and $q(\partial)$ have a common left factor $\varsigma(\partial)$.

"$\Rightarrow$" Suppose that the polynomials $p(\partial), q(\partial)$ with $\mathrm{d}^0 p = n$ and $\mathrm{d}^0 q = s$ have a common left factor $\varsigma(\partial)$ with $\mathrm{d}^0\varsigma > 0$ such that equation (17) can be written as

$$p(\partial)\mathrm{d}y - q(\partial)\mathrm{d}u = \varsigma(\partial)\left[\tilde{p}(\partial)\mathrm{d}y - \tilde{q}(\partial)\mathrm{d}u\right] = 0.$$

The one-form $\tilde{p}(\partial)\mathrm{d}y - \tilde{q}(\partial)\mathrm{d}u$ either is exact, or can be made exact by multiplying with the integrating factor. To prove this one has to show that $\omega = \tilde{p}(\partial)\mathrm{d}y - \tilde{q}(\partial)\mathrm{d}u$ satisfies the condition $\mathrm{d}\omega \wedge \omega = 0$, given the one-form $p(\partial)\mathrm{d}y - q(\partial)\mathrm{d}u$ is exact. Then

$$\tilde{p}(\partial)\mathrm{d}y - \tilde{q}(\partial)\mathrm{d}u = \mathrm{d}\varphi_r$$

and we get

$$\varsigma(\partial)\mathrm{d}\varphi_r = 0,$$

which implies the existence of F such that (18) holds. Hence the system is not irreducible.

7. EXAMPLES

Since the mathematical tools we employ require that instead of working with the equations themselves we work with their differentials, the systems $\varphi(\cdot) = 0$ and $\varphi(\cdot) + \text{const} = 0$ are not distinguished for arbitrary const. In order to avoid such situations we fix the constant and assume it to be defined by the equilibrium point of the system, around which the one-forms will be integrated to get the reduced system equations.

Example 11. Consider the dynamical system described by input-output delta-differential equation

$$y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + y^{\Delta} - uy = 0. \quad (21)$$

By applying the d operator for (21) we obtain

$$\left(\partial^2 + (1 - u^{\sigma})\partial - u^{\Delta} - u\right)\mathrm{d}y = \\ = \left(y^{\sigma}\partial + y^{\Delta} + y\right)\mathrm{d}u.$$

Applying the Euclidean division algorithm, we get

$$p(\partial) = q(\partial)\left(\frac{1}{y}\partial - \frac{u}{y}\right),$$

if only $y \neq 0$. So $q(\partial) = y^{\sigma}\partial + y^{\Delta} + y$ is the greatest common left divisor of $p(\partial) = \partial^2 + (1 - u^{\sigma})\partial - u^{\Delta} - u$ and $q(\partial)$. Then $\left(\frac{1}{y}\partial - \frac{u}{y}\right)\mathrm{d}y = \mathrm{d}u$ or alternatively $\mathrm{d}\left[y^{\Delta} - yu\right] = 0$. Hence, system (21) is reducible. The autonomous element for (21) is the function $\varphi_r(y, y^{\Delta}, u) = y^{\Delta} - yu$ and (21) can be written as $\varphi_r^{\Delta} + \varphi_r = 0$.

Example 12. Consider the dynamical system described by input-output delta-differential equation

$$y^{[2]} - y^{\Delta}u - yu^{\Delta} - \mu y^{\Delta}u^{\Delta} + yy^{\Delta} - uy^2 = 0. \quad (22)$$

By applying the d operator for (22) we obtain

$$\left(\partial^2 + (y - u^{\sigma})\partial - u^{\Delta} + y^{\Delta} - 2uy\right)\mathrm{d}y = \\ = \left(y^{\sigma}\partial + y^{\Delta} + y^2\right)\mathrm{d}u.$$

Applying the Euclidean division algorithm, we get

$$\partial^2 + (y - u^{\sigma})\partial - u^{\Delta} + y^{\Delta} - 2uy = \\ = \left(y^{\sigma}\partial + y^{\Delta} + y^2\right)\left(\frac{1}{y}\partial - \frac{u}{y}\right) + y^{\Delta} - uy,$$

so

$$p(\partial) = q(\partial)\left(\frac{1}{y}\partial - \frac{u}{y}\right) + \left(y^{\Delta} - uy\right),$$

if only $y \neq 0$. Therefore $p(\partial) = \partial^2 + (y - u^{\sigma})\partial - u^{\Delta} + y^{\Delta} - 2uy$ and $q(\partial) = y^{\sigma}\partial + y^{\Delta} + y^2$ have no common left factors. Hence, system (22) is irreducible.

8. CONCLUSIONS

In the paper we proved a necessary and sufficient condition for reducibility of a nonlinear control system on a homogenous time scale, described by the n-th order delta differential equation. In practice it means finding a common left factor of two polynomials defined over the σ-differential field of meromorphic functions. To this aim it is enough to use the left Euclidian division algorithm, extended to the Ore polynomials. The proposed condition and procedure unify the existing results concerning polynomial approach to reducibility of i/o behavior of nonlinear control system in discrete and continuous time.

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