Proceedings of Symposia in Pure Mathematics

Real Analytic Geometry and Local Observability

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Abstract. The language of real analytic geometry is used to give necessary and sufficient conditions of stable local observability of analytic systems. A full proof is provided in the most general case. Globalization of this result yields a new characterization of local observability at large which uses the concept of sheaf.

1. Introduction

The modern real analytic geometry started 50 years ago with the fundamental paper of Henri Cartan [**Ca**]. It was the first time that essential differences between real and complex analytic geometries were discovered (e.g lack of coherence in the real case). The language of germs and sheaves, algebras and ideals, commonly used in modern geometry, nicely fits in the study of observability of real analytic (control) systems. We use it to express and prove a characterization of stable local observability at a distinguished point, which leads to a new necessary and sufficient condition of local observability at large. This is a continuation of work started in papers of D. Mozyrska and mine [**Mo, MB1, MB2**], where preliminary results were presented under some regularity assumptions.

In Section 2 we recall basic definitions and facts from real analytic geometry. In particular, the interplay between geometry and algebra is presented in detail as we are going to use it extensively throughout the paper. Real radical, one of the main tools to be exploited in the paper, is of special interest as it exhibits the peculiarities of real geometry.

Section 3 contains the main result, a characterization of stable local observability, which is preceded by necessary definitions and a short review of known facts. Stable local observability was introduced in [**Ba2**]. It holds at a point x_0 if the system is locally observable at every point of some neighborhood of x_0.

We globalize the main result in Section 4 using real analytic sets and corresponding sheaves of ideals of germs of real analytic functions. This leads to a new characterization of local observability understood globally, i.e. at every point. We compare this with our former criterion [**Ba1**].

1991 *Mathematics Subject Classification.* Primary 93B07, 14P15; Secondary 93B29, 93B27.

This work was supported by KBN grant No. 8 T11A 011 12.

The proof of the main result is contained in Section 5. It is complete and without additional assumptions of regularity. In particular, coherence of the constructed ideals, long thought by the author to be essential in the proof, need not be assumed. Though this strips the theory off a potential deepness, but the theorem will definitely be easier to use in practice.

2. Basic real geometry

We assume that the reader is familiar with the concepts of germs of functions and sets, and with fundamentals of the sheaf theory and theory of analytic sets. Necessary definitions can be found e.g. in [**GR, GMT**]. If B is a "global" object (a set, a function, or a family of functions), B_x will always denote its germ at the point x (but the precise meaning of the germ will depend on the meaning of the object). If C is a germ, $\tilde{C}$ will denote one of its representatives. By $\mathcal{O}_x$ we denote the algebra of germs of real analytic functions at x, where $x \in \mathbb{R}^n$ (n fixed throughout the paper), and by m_x the (only) maximal ideal in $\mathcal{O}_x$, consisting of all germs in $\mathcal{O}_x$ that vanish at x. By $\mathcal{O}$ we denote the sheaf of germs of real analytic functions on $\mathbb{R}^n$.

If U is an open subset in $\mathbb{R}^n$ then $\mathcal{O}_U$ will mean the algebra of real analytic functions on U. Thus $U \mapsto \mathcal{O}_U$ is the presheaf of partially defined real analytic functions on $\mathbb{R}^n$. If A is a subalgebra of $\mathcal{O}_U$ and $x_0 \in U$ then A_{x_0} means the set of germs at x_0 of functions from A. Of course A_{x_0} is again an algebra over $\mathbb{R}$. If I is an ideal in $\mathcal{O}_U$ then I_{x_0} means the ideal in $\mathcal{O}_{x_0}$ generated by the germs at x_0 of function from I.

Consider a set-germ A in $\mathbb{R}^n$ (at some point x). Then $J(A)$ denotes the ideal in $\mathcal{O}_x$ of germs (at x) of real analytic functions that vanish on A. If I is an ideal in $\mathcal{O}_x$ then $Z(I)$ will denote the zero set-germ of I (at x). Let us recall that $Z(I)$ is defined as the intersection of the set-germs $Z(\varphi_i)$, $i = 1, \ldots, k$, where $\varphi_1, \ldots, \varphi_k$ are generators of the ideal I. Since only finite intersections of set-germs are defined, we must use here the property that $\mathcal{O}_x$ is Noetherian.

We have a natural duality between ideals and set-germs. If $I_1 \subset I_2$ then $Z(I_2) \subset Z(I_1)$

Let P be any commutative ring with a unit and I be an ideal in P. Then the *real radical of* I, denoted by $\sqrt[\mathbb{R}]{I}$, is the set of all $a \in P$ for which there is $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $b_1, \ldots, b_k \in P$ such that

$$a^{2m} + b_1^2 + \ldots + b_k^2 \in I.$$

The real radical is an ideal in P and it contains I. If I is a proper ideal in P, then also $\sqrt[\mathbb{R}]{I}$ is proper. An ideal I is called *real* if $\sqrt[\mathbb{R}]{I} = I$.

THEOREM 2.1 ([**Ri**]). *Let $x \in \mathbb{R}^n$. If I is an ideal in $\mathcal{O}_x$ then $J(Z(I)) = \sqrt[\mathbb{R}]{I}$.* □

Theorem 2.1 implies that there is a 1:1 correspondence between germs of analytic sets at x and real ideals in $\mathcal{O}_x$.

Let $\mathcal{G}$ be a family of analytic functions on some open set $U \subset \mathbb{R}^n$. Denote by $S_x(\mathcal{G})$ the germ at x of the level set of $\mathcal{G}$ that passes through x. In other words

$$S_x(\mathcal{G}) = \{y \in U : \varphi(x) = \varphi(y) \text{ for } \varphi \in \mathcal{G}\}_x.$$

The set-germ $S_x(\mathcal{G})$ is a germ of analytic set. One of the representatives of $S_x(\mathcal{G})$ is the analytic set in U: $\{y \in U : \varphi(x) = \varphi(y) \text{ for } \varphi \in \mathcal{G}\}$.

3. Observability

Let us consider a control system with output

$$(\Sigma)\quad \begin{array}{rcl} \dot{x} & = & f(x,u) \\ y & = & h(x), \end{array} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^r$ and $u(t) \in \Omega$ - arbitrary set. If $\omega \in \Omega$ then f_ω, defined by $f_\omega(x) = f(x,\omega)$, is a vector field on $\mathbb{R}^n$. We assume that the map h and the vector field f_ω for every $\omega \in \Omega$ are analytic, and that controls u are piecewise constant functions of time. By $\gamma(t, x_0, u)$ we denote the solution of the equation $\dot{x} = f(x,u)$ corresponding to control u and the initial condition $x(0) = x_0$, and evaluated at time t.

Let $x_1, x_2 \in \mathbb{R}^n$. Then they are *indistinguishable* if

$$h(\gamma(t, x_1, u)) = h(\gamma(t, x_2, u))$$

for every control u and every $t \geq 0$ for which both sides of the equation are defined. Otherwise x_1 and x_2 are *distinguishable.*

We say that Σ is *locally observable at* x_0 (LO(x_0)) if there is a neighborhood U of x_0 such that for every $x \in U$, x and x_0 are distinguishable. We say that Σ is *stably locally observable at* x_0 (SLO(x_0)) if there is a neighborhood U of x_0 such that Σ is locally observable at x for every $x \in U$. We call Σ *locally observable* (LO) or *stably locally observable* (SLO) when it is locally observable or stably locally observable at every $x \in \mathbb{R}^n$.

By the observation algebra of system Σ, denoted by $\mathcal{H}(\Sigma)$, we mean the smallest algebra over $\mathbb{R}$ of real analytic functions on $\mathbb{R}^n$ containing the components of the map h and closed under the Lie derivatives with respect to the vector fields f_ω, $\omega \in \Omega$.

Let us recall the main results concerning local observability. From here, whenever Σ is fixed, we replace $\mathcal{H}(\Sigma)$ by $\mathcal{H}$.

Theorem 3.1 ([**HK**]).
a) Points x_1 and x_2 are indistinguishable iff $\varphi(x_1) = \varphi(x_2)$ for every $\varphi \in \mathcal{H}$.
b) If $\dim d\mathcal{H}(x_0) = n$ then Σ is LO(x_0).
c) ($\forall x_0 \in \mathbb{R}^n$: $\dim d\mathcal{H}(x_0) = n$) $\Rightarrow$ ($\forall x_0 \in \mathbb{R}^n$: Σ is LO(x_0)) $\Rightarrow$
($\exists X$ - a real analytic set in $\mathbb{R}^n$: $\forall x_0 \notin X$: $\dim d\mathcal{H}(x_0) = n$). □

Let us denote the condition $\dim d\mathcal{H}(x_0) = n$ by HK(x_0) (Hermann-Krener condition at x_0). The last part of Theorem 3.1 says that if we are interested in local observability at large, i.e. at each point, then condition HK(x_0) is satisfied almost everywhere, so the gap between sufficient condition and necessary condition for local observability at large is quite narrow. However, when one is interested in local observability at a particular point of the state space, the Hermann-Krener condition may be far from being necessary (see [**Ba1**]).

We say that $\mathcal{H}$ is *injective at* x_0 if there is a neighborhood U of x_0 such that for every $x_1, x_2 \in U$: the condition $\varphi(x_1) = \varphi(x_2)$ for every $\varphi \in \mathcal{H}$ implies $x_1 = x_2$. Then we have a slightly stronger statement.

Proposition 3.2. *HK(x_0) $\Rightarrow$ $\mathcal{H}$ is injective at x_0 $\Rightarrow$ Σ is SLO(x_0) $\Rightarrow$ Σ is LO(x_0).* □

None of the implications in Proposition 3.2 may be, in general, reversed. But we have important, though obvious, global equivalence.

PROPOSITION 3.3. *Σ is SLO(x_0) for every $x_0 \in \mathbb{R}^n$ iff Σ is LO(x_0) for every $x_0 \in \mathbb{R}^n$. In other words, Σ is LO iff Σ is SLO.* □

Let J_x be the ideal in $\mathcal{O}_x$ generated by germs at x_0 of functions from $\mathcal{H}$ that vanish at x.

THEOREM 3.4 ([**Ba1**]).
a) Σ is LO(x_0) iff $\sqrt[\mathbb{R}]{J_{x_0}} = m_{x_0}$.
b) HK(x_0) iff $J_{x_0} = m_{x_0}$. □

If G is a subset of $\mathcal{O}_x$ then $D(G)$ will denote the ideal in $\mathcal{O}_x$ generated by Jacobians $\frac{\partial(\varphi_1,\ldots,\varphi_n)}{\partial(x_1,\ldots,x_n)}$, where $\varphi_i \in G$. Observe that these Jacobians are well defined on germs of functions. If $\mathcal{G}$ is a family of real analytic functions on an open set U in $\mathbb{R}^n$ then similarly we define the Jacobian ideal $D(\mathcal{G})$ in $\mathcal{O}_U$. We shall need also extensions of these concepts to the case where x belongs to an analytic manifold M and U is an open set in M with a global coordinate system (i.e. diffeomorphic to an open subset of $\mathbb{R}^n$). Though Jacobians will then depend on a chosen coordinate system, Jacobian ideals $D(G)$ or $D(\mathcal{G})$ will not. Furthermore, there is a simple relation between Jacobian ideals for functions and germs of functions. If $\mathcal{G}$ is a family of analytic functions on U and $x_0 \in U$ then we have

$$D(\mathcal{G})_{x_0} = D(\mathcal{G}_{x_0}). \tag{3.2}$$

Now, for a point $x \in \mathbb{R}^n$, we define a sequence of ideals in $\mathcal{O}_x$. Let $I_x^{(0)} = (0)$ and $I_x^{(k+1)} = \sqrt[\mathbb{R}]{D(\mathcal{H}_x \cup I_x^{(k)})}$. It is clear that instead of $\mathcal{H}_x$ in this definition one can take previously defined ideal J_x, i.e. $I_x^{(k+1)} = \sqrt[\mathbb{R}]{D(J_x \cup I_x^{(k)})}$. This leads to the following generalization of statement b) of Theorem 3.4.

COROLLARY 3.5. *HK(x_0) $\Leftrightarrow J_{x_0} = m_{x_0} \Leftrightarrow I_{x_0}^{(1)} = \mathcal{O}_{x_0}$.* □

The ideals $I_x^{(k)}$ will be the main tools in studying stable local observability. First we prove the basic fact.

PROPOSITION 3.6. *For any $k \geq 0$, $I_x^{(k)} \subset I_x^{(k+1)}$, and there is $s \geq 0$ such that $I_x^{(s)} = I_x^{(s+1)}$.*

PROOF. To prove the first part we proceed by induction. It is clear that $I_x^{(0)} \subset I_x^{(1)}$, so assume that $I_x^{(k)} \subset I_x^{(k+1)}$ for some $k \geq 0$. Then also $\mathcal{H}_x \cup I_x^{(k)} \subset \mathcal{H}_x \cup I_x^{(k+1)}$ and $D(\mathcal{H}_x \cup I_x^{(k)}) \subset D(\mathcal{H}_x \cup I_x^{(k+1)})$, so finally $\sqrt[\mathbb{R}]{D(\mathcal{H}_x \cup I_x^{(k)})} \subset \sqrt[\mathbb{R}]{D(\mathcal{H}_x \cup I_x^{(k+1)})}$. This means that $I_x^{(k+1)} \subset I_x^{(k+2)}$. Since the ring $\mathcal{O}_x$ is Noetherian the sequence of ideals $I_x^{(k)}$ must stabilize at some s. □

Now we can state the main result of the paper.

THEOREM 3.7. *System Σ is SLO(x_0) iff $I_{x_0}^{(s)} = \mathcal{O}_{x_0}$ for some $s > 0$.*

The proof of Theorem 3.7 is contained in Section 5.

The statements of Corollary 3.5, Proposition 3.6 and Theorem 3.7 can be translated into the language of germs of analytic sets. Let $X_x^{(k)} = Z(I_x^{(k)})$. Because the ideals $I_x^{(k)}$ are real we also get $J(X_x^{(k)}) = I_x^{(k)}$, so there is one-to-one correspondence between the ideals and the set-germs. We have then the following:

Corollary 3.8. *Let* $x_0 \in \mathbb{R}^n$.
a) HK(x_0) *iff* $X_{x_0}^{(1)} = \emptyset$.
b) For every $k \geq 0$, $X_{x_0}^{(k)} \supset X_{x_0}^{(k+1)}$, *and for some* $s \geq 0$, $X_{x_0}^{(s)} = X_{x_0}^{(s+1)}$.
c) Σ *is SLO*(x_0) *iff* $X_{x_0}^{(s)} = \emptyset$. □

Example 3.9 ([**MB2**]). Consider the following system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ \dot{x}_3 &= 0 \\ y &= x_1^2 + x_2^2 + x_3^2 \end{aligned}$$

and choose $x_0 = 0$. Then $\mathcal{H} = \mathbb{R}[x_1^2, x_2^2, x_3^2]$ and $I_{x_0}^{(1)} = J_{x_0} = (x_1x_2x_3)$. So $X_{x_0}^{(1)}$ is the germ of a union of three planes intersecting at 0. In the next step we obtain $I_{x_0}^{(2)} = \sqrt[\mathbb{R}]{(x_1^2x_2^2, x_2^2x_3^2, x_1^2x_3^2)} = (x_1x_2, x_2x_3, x_1x_3)$. A quick calculation shows that X_{x_0} is the germ of a union of three lines intersecting at 0. Finally, $I_{x_0}^{(3)} = m_{x_0}$ and then, inevitably, $I_{x_0}^{(4)} = \mathcal{O}_{x_0}$, so Σ is stably locally observable at x_0.

4. Globalization

Using the language of sheaves we can nicely sew up local characterizations of stable local observability into a global criterion. Let $I^{(k)}$ be a sheaf of ideals in $\mathcal{O}$ whose stalk at $x \in \mathbb{R}^n$ is equal $I_x^{(k)}$. In other words, we put $I^{(0)} = (0)$ - the sheaf of zero-ideals, and define $I^{(k)}$ inductively by

$$I^{(k+1)} = \sqrt[\mathbb{R}]{D(\mathcal{H} \cup I^{(k)})}$$

where $\mathcal{H}$ is identified with the set of sections of the sheaf $x \mapsto \mathcal{H}_x$ and operations D and $\sqrt[\mathbb{R}]{}$ are performed germwise.

Each sheaf of ideals corresponds to an analytic set. So we can define the set $X^{(k)}$ as follows:

$$x \in X^{(k)} \Leftrightarrow I_x^{(k)} \neq \mathcal{O}_x.$$

It's clear that the germ of $X^{(k)}$ at x is precisely $X_{x_0}^{(k)}$.

Since local observability and stable local observability are equivalent, we have a new criterion of local observability. It's a consequence of Theorem 3.7.

Theorem 4.1. Σ *is LO iff* $I^{(s)} = \mathcal{O}$ *for some* $s > 0$. □

Remark 4.2. Part a) of Theorem 3.4 gives also necessary and sufficient condition of local observability at large. Namely: Σ is LO iff $\forall x \in \mathbb{R}^n$, $\sqrt[\mathbb{R}]{J_x} = m_x$. However neither the ideals J_x nor m_x form a sheaf, so we have to check this condition at every point separately.

Translating ideals into sets we obtain the following:

Corollary 4.3. Σ *is LO iff* $X^{(k)} = \emptyset$ *for some* $k > 0$. □

Example 4.4. Let the observation algebra of a system Σ on $\mathbb{R}^3$ be generated by 3 functions: $\varphi_1 = x_1x_3 + x_2^2/2$, $\varphi_2 = x_1^2/2 + x_2^2/2$ and $\varphi_3 = x_2(x_1 - x_3)$. Then $I^{(1)} = \sqrt[\mathbb{R}]{D(\{\varphi_1, \varphi_2, \varphi_3\})} = \sqrt[\mathbb{R}]{(x_3(x_1^2 + x_2^2) - x_1^3)}$. The set $X^{(2)}$ is the famous Cartan's umbrella which serves as a simple example of a lack of coherence (see [**Ca, GMT**]). In the next step we get $I^{(2)} = (x_2, x_1(x_1 - x_3))$. The corresponding set is a union of two lines. The ideals stabilize at $k = 3$: $I^{(3)} = (x_1, x_2) = I^{(4)}$ and $X^{(3)}$ is a line - the stick of the umbrella - consisting of singular points of $X^{(1)}$. Thus Σ is not locally observable.

5. Proof of Theorem 3.7

Sufficiency.

LEMMA 5.1. *Let $U \subset \mathbb{R}^n$ be an open set and X be an analytic set in U. Consider a family $\mathcal{G}$ of analytic functions on U. If $\forall x \notin X$: $S_x(\mathcal{G}) = \{x\}$ then $\forall x \in X$: $S_x(\mathcal{G}) \subset X_x$.*

PROOF. Suppose that there is $x \in X$ such that $S_x(\mathcal{G}) \not\subset X_x$. This means that for every representatives $\tilde{S}_x(\mathcal{G})$ and $\tilde{X}_x$ we have $\tilde{S}_x(\mathcal{G}) \not\subset \tilde{X}_x$. Take $\tilde{X}_x := X$ and arbitrarily small neighborhood V of x in U. Let $\tilde{S}_x(\mathcal{G})$ be a representative of $S_x(\mathcal{G})$ in V. Then there is $y \in \tilde{S}_x(\mathcal{G})$ such that $y \notin X$. Take a sequence (y_n) of such points converging to x. We may assume that all these points belong to some (large enough) representative Y of $S_x(\mathcal{G})$ and that Y is an analytic set. Only finite number of points y_n may be isolated points of Y. This means that arbitrarily close to x there is a point y_n for which $S_{y_n}(\mathcal{G}) \neq \{y_n\}$. Thus we get a contradiction. □

LEMMA 5.2. *Let U be an open set in $\mathbb{R}^n$ and $\mathcal{G}$ be a family of analytic functions on U. For every $x \in U$: if $x \notin Z(D(\mathcal{G}))$ then $S_x(\mathcal{G}) = \{x\}$.*

PROOF. If $x \notin Z(D(\mathcal{G}))$ then there are functions $\varphi_1, \ldots, \varphi_n \in G$ such that

$$\frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}(x) \neq 0.$$

Thus the map $\bar{x} \mapsto (\varphi_1(\bar{x}), \ldots, \varphi_n(\bar{x}))$ is injective in a neighborhood of x. This implies that $S_x(\mathcal{G}) = \{x\}$. □

LEMMA 5.3. *Assume that $X_{x_0}^{(k)} \neq X_{x_0}^{(k+1)}$ for some $k \geq 0$. Then there is a neighborhood U of x_0 in $\mathbb{R}^n$ and a representative $\tilde{X}_{x_0}^{(k+1)}$ of $X_{x_0}^{(k+1)}$ in U such that for every $x \in U$:*

$$x \notin \tilde{X}_{x_0}^{(k+1)} \Rightarrow S_x(\mathcal{H}) = \{x\}. \tag{5.1}$$

PROOF. First observe that $Z(D(\mathcal{H}_{|U}))$ is a representative of $Z(D(\mathcal{H}_{x_0}))$. We procede by induction. Let $k = 0$ and assume that $X_{x_0}^{(1)} \neq X_{x_0}^{(0)} = \mathbb{R}_{x_0}^n$. Take any neighborhood U of x_0. Since $X_{x_0}^{(1)} = Z(D(\mathcal{H}_{x_0}))$ then $Z(D(\mathcal{H}_{|U})) =: \tilde{X}_{x_0}^{(1)}$ is a representative of $X_{x_0}^{(1)}$ in U. If $x \notin \tilde{X}_{x_0}^{(1)}$ then, by Lemma 5.2, $S_x(\mathcal{H}) = S_x(\mathcal{H}_{|U}) = \{x\}$.

Now assume that the statement of the theorem holds for $k - 1 \geq 0$. Hence, there is a neighborhood U of x_0 and a representative $\tilde{X}_{x_0}^{(k)} = Z(\varphi_1, \ldots, \varphi_s)$ of $X_{x_0}^{(k)}$ such that if $x \in U$ and $x \notin \tilde{X}_{x_0}^{(k)}$ then $S_x(\mathcal{H}) = \{x\}$. The functions $\varphi_1, \ldots, \varphi_s$ are representatives on U of generators of $I_{x_0}^{(k)}$. The set $\tilde{X}_{x_0}^{(k+1)} := Z(D(\mathcal{H}_{|U} \cup \{\varphi_1, \ldots, \varphi_s\}))$ is a representative of $X_{x_0}^{(k+1)}$. Clearly $\tilde{X}_{x_0}^{(k+1)} \subset \tilde{X}_{x_0}^{(k)}$. Take $x \in U$ such that $x \notin \tilde{X}_{x_0}^{(k+1)}$. If $x \notin \tilde{X}_{x_0}^{(k)}$ then $S_x(\mathcal{H}) = \{x\}$. Assume then that $x \in \tilde{X}_{x_0}^{(k)}$. From Lemma 5.1 we get $S_x(\mathcal{H}) \subset (\tilde{X}_{x_0}^{(k)})_x$. Then $S_x(\mathcal{H}) = S_x(\mathcal{H}) \cap (\tilde{X}_{x_0}^{(k)})_x = S_x(\mathcal{H}_{|U} \cup \{\varphi_1, \ldots, \varphi_s\})$. Since $x \notin Z(D(\mathcal{H}_{|U} \cup \{\varphi_1, \ldots, \varphi_s\}))$ then, by Lemma 5.2,

$S_x(\mathcal{H}) = S_x(\mathcal{H}_{|U} \cup \{\varphi_1, \ldots, \varphi_s\}) = \{x\}$. This finishes the inductive step of the proof. □

To prove sufficiency assume that $I^{(s)}_{x_0} = \mathcal{O}_{x_0}$ for some $s > 0$. This means that $X^{(s)}_{x_0} = \emptyset$. From Lemma 5.3 it follows that nontrivial set-germs $S_x(\mathcal{H})$ (i.e. different from $\{x\}$) may be found only in $\tilde{X}^{(s)}_{x_0}$ — some representative of $X^{(s)}_{x_0}$. But $X^{(s)}_{x_0} = \emptyset$ so in some neighborhood of x_0 the level set-germs of $\mathcal{H}$ must be trivial. This means that Σ is stably locally observable at x_0.

Necessity.

Lemma 5.4. *Let M be a real analytic manifold, $x_0 \in M$, U be a neighborhood of x_0 diffeomorphic to an open set in $\mathbb{R}^n$, and $\mathcal{G}$ be a family of analytic functions on U. If $D(\mathcal{G}) = 0$ (zero ideal) then arbitrarily close to x_0 there is $x \in U$ such that $S_x(\mathcal{G}) \neq \{x\}$.*

Proof. We may assume that $M = \mathbb{R}^n$ and $U \subset \mathbb{R}^n$. Let

$$s = \max_{\substack{x \in U \\ \varphi_i \in \mathcal{G}}} \operatorname{rank} \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}(x).$$

Then $s < n$ and arbitrarily close to x_0 there is $x \in U$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{G}$ such that $\operatorname{rank} \frac{\partial(\varphi_1, \ldots, \varphi_n)}{\partial(x_1, \ldots, x_n)}(\bar{x}) = s$. This rank is preserved in some neighborhood V of $\bar{x}$. Thus we may assume that gradients of $\varphi_1, \ldots, \varphi_s$ are linearly independent at every point of V and span $d\mathcal{G}(x)$ for $x \in V$. Then by Frobenius Theorem V is a union of integral manifolds of codistribution $d\mathcal{G}$. The integral manifolds are the level sets of $\mathcal{G}$ and have dimension greater than or equal 1. This means that $S_{\bar{x}}(\mathcal{G}) \neq \{\bar{x}\}$. □

Lemma 5.5. *Let U be an open subset of $\mathbb{R}^n$ and $\varphi_1, \ldots, \varphi_k$ be analytic functions on U. Assume that $Y = Z(\varphi_1, \ldots, \varphi_k)$ is an analytic manifold in U. Let $x_0 \in Y$ and $\mathcal{G}$ be a family of analytic functions on U.*

If $Y \subset Z(D(\mathcal{G} \cup \{\varphi_1, \ldots, \varphi_k\}))$ then arbitrarily close to x_0 there is $x \in Y$ such that $S_x(\mathcal{G}) \neq \{x\}$.

Proof. We may assume that $\nabla\varphi_1, \ldots, \nabla\varphi_k$ are linearly independent at every point of U. Changing the coordinates we can obtain $\varphi_i = x_i$, $i = 1, \ldots, k$. Let $\psi_1, \ldots, \psi_{n-k} \in \mathcal{G}$. Then for $x \in Y$ we have

$$0 = \det \begin{bmatrix} \nabla\varphi_1(x) \\ \vdots \\ \nabla\varphi_k(x) \\ \nabla\psi_1(x) \\ \vdots \\ \nabla\psi_{n-k}(x) \end{bmatrix} = \det \begin{bmatrix} I \quad 0 \\ \left(\frac{\partial\psi_i}{\partial x_j}(x)\right)_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,n}} \end{bmatrix}$$

$$= \det \left(\frac{\partial\psi_i(x)}{\partial x_j}\right)_{\substack{i=1,\ldots,n-k \\ j=k+1,\ldots,n}}.$$

Hence, after restricting to manifold Y, we get $D(\mathcal{G}_{|Y}) = 0$. From Lemma 5.4, arbitrarily close to x_0 there is $x \in Y$ such that $S_x(\mathcal{G}_{|Y}) \neq \{x\}$. But $S_x(\mathcal{G}_{|Y}) \subset S_x(\mathcal{G})$ so also $S_x(\mathcal{G}) \neq \{x\}$. □

LEMMA 5.6. *Assume that* $X_{x_0}^{(s)} = X_{x_0}^{(s+1)}$ *for some* $s \geq 0$ *and* $X_{x_0}^{(s)} \neq \emptyset$. *Then in every neighborhood of* x_0 *there is* x *such that* $S_x(\mathcal{H}) \neq \{x\}$.

PROOF. Let $\varphi_1, \ldots, \varphi_k$ be representatives of generators of ideal $I_{x_0}^{(s)}$, defined on some common neighborhood U of x_0. Then $\tilde{X}_{x_0}^{(s)} := Z(\varphi_1, \ldots, \varphi_k)$ is a representative of $X_{x_0}^{(s)}$ in U. In every neighborhood of x_0 one can find a regular point of the analytic set $\tilde{X}_{x_0}^{(s)}$ (see [**Na, Ca**]). Let $\bar{x}$ be such a point and let V be a neighborhood of $\bar{x}$ in $\mathbb{R}^n$ such that $Y := V \cap \tilde{X}_{x_0}^{(s)}$ is an analytic manifold. Then $Y = Z(\varphi_{1|V}, \ldots, \varphi_{k|V})$. Let $x \in Y$. Then $x \in \tilde{X}_{x_0}^{(s+1)} := Z(D(\mathcal{H}_{|U} \cup \{\varphi_1, \ldots, \varphi_k\}))$ so $Y \subset Z(D(\mathcal{H}_{|V} \cup \{\varphi_{1|V}, \ldots, \varphi_{k|V}\}))$. The statement of the lemma follows now from Lemma 5.5. □

Lemma 5.6 shows that if $\tilde{X}_{x_0}^{(s)} = \tilde{X}_{x_0}^{(s+1)}$ for some $s \geq 0$ then Σ is not stably locally observable.

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