Rank condition of stable local observability of analytic systems on $\mathbb{R}^3$

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1 Introduction

In this paper we formulate and prove a necessary and sufficient condition of stable local observability of analytic systems on $\mathbb{R}^3$. It is expressed by a rank condition over the ring of germs of analytic functions.

Let Σ be an analytic control system on $\mathbb{R}^n$ given by the equations

$$\dot{x} = f_0(x) + \Sigma_{i=1}^m u_i f_i(x) \quad (1)$$
$$y = h(x). \quad (2)$$

We assume that $u(t)$ is an element of an open set $\Omega \subset \mathbb{R}^m$ and we take only piecewise constant controls u; h is a C^ω map from $\mathbb{R}^n$ to $\mathbb{R}^r$.

By $\gamma(t, x_0, u)$ we denote the solution of (1) corresponding to the initial condition $\gamma(0) = x_0$ and control u, and evaluated at time t.

We say that $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* (with respect to Σ) if

$$h(\gamma(t, x_1, u)) = h(\gamma(t, x_2, u)), \quad (3)$$

for every control u and for every time $t \geq 0$, for which both sides of (3) are defined. Otherwise, x_1 and x_2 are *distinguishable* (with respect to Σ).

We say that Σ is *locally observable* at $x_0 \in \mathbb{R}^n$ if there is a neighborhood U of x_0 such that for every $x \in U$, x_0 and x are distinguishable.

We say that Σ is *stably locally observable* at x_0 if there is an open neighborhood V of x_0 such that for every $x \in V$, Σ is locally observable at x_0. It means that local observability is preserved when the initial state is slightly changed or when it is not known exactly.

By the *observation algebra* of the system Σ we mean the smallest subalgebra of $C^\omega(\mathbb{R}^n, \mathbb{R})$ containing all components of h and closed under Lie derivatives with respect to vector fields of Σ. It is denoted by $H(\Sigma)$.
The observation algebra $H(\Sigma)$ is generated by the set $\mathcal{H}(\Sigma)$ consisting of functions of the form $L_{f_{i_k}} \dots L_{f_{i_1}} h_j$, where $j = 1, \dots, r$; $k \geq 0$; $i_s = 0, \dots m$.

Let $\mathcal{O}_x$ denote the algebra over $\mathbb{R}$ of germs at x of C^ω functions on $\mathbb{R}^n$. Define m_x to be the maximal ideal of $\mathcal{O}_x$. It consists of all the germs that vanish at x.
By I_x we mean the ideal of $\mathcal{O}_x$ generated by germs of those functions from $H(\Sigma)$ which vanish at x ($I_x \subset m_x$).
Let $\sqrt[\mathbb{R}]{I}$ denote the *real radical* of an ideal I in a commutative ring R. It is defined as the set of all elements $a \in R$ such that there are integers $m > 0$, $k \geq 0$ and $b_1, b_2, \dots, b_k \in R$ such that $a^{2m} + b_1^2 + \dots + b_k^2 \in I$.

A necessary and sufficient condition of local observability was proved by Z. Bartosiewicz in [1].

Theorem 1.1 *The following conditions are equivalent:*
(i) $\sqrt[\mathbb{R}]{I_x} = m_x$,
(ii) Σ *is locally observable at* x. □

Now we are interested in conditions of stable local observability. We know from [2] that Hermann-Krener condition is stable.

Theorem 1.2 *If* $\dim dH(\Sigma, x_0) = n$ *then local observability at* x_0 *is stable.* $\square$

A necessary and sufficient condition of stable local observability of analytic system in geometric form was proved by Z. Bartosiewicz in [3]. In the next section we give an algebraic condition which should be easier to check.

2 Main result

Let $x_0 \in U_{x_0} \subset \mathbb{R}^n$. By $\mathcal{H}_0 := \mathcal{H}(\Sigma)_{x_0}$ we mean the set of germs at x_0 of analytic functions from $\mathcal{H}(\Sigma)$. For simplicity we assume here that $\mathcal{H}_0$ is finite, so we may write

$$\mathcal{H}_0 = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_m \end{pmatrix}.$$

Now we define a set-germ at x_0

$$X_1 = \{x \in U_{x_0} : \operatorname{rank} \mathcal{H}_0'(x) < n \}_{x_0},$$

where in the matrix

$$\mathcal{H}_0'(x) = \begin{pmatrix} \frac{\partial \varphi_1}{\partial x}(x) \\ \vdots \\ \frac{\partial \varphi_m}{\partial x}(x) \end{pmatrix}$$

we use the representatives of germs from $\mathcal{H}_0$.
If $X_1 \neq \emptyset$, let $\mathcal{H}_1 := (\mathcal{H}_0)_{|X_1}$.
Inductively we define
$X_{k+1} :=$

$$\{x \in X_k : \operatorname{rank} \mathcal{H}_k'(x) < \dim X_k\}_{x_0},$$

and $\mathcal{H}_{k+1} := (\mathcal{H}_k)_{|X_{k+1}}$. In this definition we assume that all the set-germs X_k are germs of manifolds. By $I(X_k)$ we denote the ideal in $\mathcal{O}_{x_0}$ of function-germs vanishing on X_k. Since X_k is a germ of a manifold, we can find a finite set of generators: $I(X_k) = (g_1, \ldots, g_p)$, $1 \leq p \leq n$. Whenever we choose generators we assume that $g_i \notin (g_1, \ldots, g_{i-1})$. Observe that X_k is a germ of a manifold iff x_0 is a regular point of X_k iff rank $\begin{pmatrix} g_1' \\ \vdots \\ g_p' \end{pmatrix} = p$.

Let us consider the situation in two-dimensional case. Now $x_0 \in \mathbb{R}^2$, $x_0 = (x_{01}, x_{02})$, and we can have only two sets - X_1, X_2. Then

$$I(X_1) = \begin{cases} (g_{11}) & \text{if } \dim X_1 = 1 \\ (g_{11}, g_{12}) & \text{if } \dim X_1 = 0 \end{cases}.$$

If $\dim X_1 = 2$, then $g_{11} = 0$.
Observe that, from [6],

$$I(X_1) = \sqrt[\mathbb{R}]{(\det_i \mathcal{H}_0')},$$

where $(\det_i \mathcal{H}_0')$ denotes the ideal in $\mathcal{O}_{x_0}$ generated by all 2×2 minors of $\mathcal{H}_0'$.
Let $\mathcal{O}_2$ denote the algebra over $\mathbb{R}$ of germs at point $x_0 = (x_{01}, x_{02})$ of C^ω functions of two variables. Then by $\mathcal{O}_{X_1}$ we denote the quotient ring $\mathcal{O}_{X_1} := \mathcal{O}_2 / I(X_1)$.
Let $g = g_{11}$ if $\dim X_1 = 1$ and $g = \begin{pmatrix} g_{11} \\ g_{12} \end{pmatrix}$ if $\dim X_1 = 0$.

The following result for system Σ on $\mathbb{R}^2$ was proved in [5].

Theorem 2.1 *Assume that* X_1 *is a germ of a manifold. System* Σ *is stably locally observable at* $x_0 \in \mathbb{R}^2$ *iff*

$$\operatorname{rank}_{\mathcal{O}_{X_1}} \begin{pmatrix} \mathcal{H}_0'/I(X_1) \\ g'/I(X_1) \end{pmatrix} = 2. \ \square$$

Remark 2.1 The condition of Theorem 2.1 means that in some neighborhood U_{x_0} of x_0, rank $\begin{pmatrix} \mathcal{H}_0'(x) \\ g'(x) \end{pmatrix} = 2$ for every $x \in U_{x_0} \backslash \{x_0\}$. This rank may drop at x_0. $\square$

We are going now to consider stable local observability of analytic systems on $\mathbb{R}^3$.
Let $\mathcal{O}_3$ denote the algebra over $\mathbb{R}$ of germs at point $x_0 = (x_{01}, x_{02}, x_{03})$ of C^ω functions of three variables. Let $U_{x_0} \subset \mathbb{R}^3$ denote the neighborhood of x_0. Define

$$X_1 = \{x \in U_{x_0} : \operatorname{rank} \mathcal{H}_0'(x) < 3 \}_{x_0}.$$

then again $I(X_1) = \sqrt[\mathbb{R}]{(\det_i \mathcal{H}_0')}$.
We have three possibilities: $I(X_1) = g_1$ or $I(X_1) = (g_1, g_2)$ or $I(X_1) = (g_1, g_2, g_3)$.

Corrolary 2.1 *If* $I(X_1) = (g_1, g_2, g_3)$ *then* Σ *is stably locally observable.*

Proof: $I(X_1) = (g_1, g_2, g_3)$ means that $X_1 = \{x_0\}$. Then from [3] Σ is stably locally observable. $\square$

Denote $g_{X_1} = g_1$ or $g_{X_1} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ and define

$$X_2 = \{x \in X_1 : \operatorname{rank} \begin{pmatrix} \mathcal{H}_0'(x) \\ g_{X_1}'(x) \end{pmatrix} < 3\}.$$

(One will see later that this is consistent with the previous general definition on $\mathbb{R}^n$.) Then

$$I(X_2) = \sqrt[\mathbb{R}]{(\det{}_i \begin{pmatrix} \mathcal{H}_0' \\ g_{X_1}' \end{pmatrix})}.$$

If $X_2 \neq X_1$ then $I(X_2) = (g_1, g_2)$ or $I(X_2) = (g_1, g_2, g_3)$.

Denote $g_{X_2} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ or $g_{X_2} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix}$ and $\mathcal{O}_{X_2} = \mathcal{O}_3 / I(X_2)$.

Now we can state the main result of the paper.

Theorem 2.2 *Assume that X_1 and X_2 are germs of manifolds. System Σ is stably locally observable at x_0 iff $I(X_2)$ is generated by at least two generators and* $\operatorname{rank}_{\mathcal{O}_{X_2}} \begin{pmatrix} \mathcal{H}_0'/I(X_2) \\ g_{X_2}'/I(X_2) \end{pmatrix} = 3.$

Example 2.1 Let Σ be the system

$$\dot{x}_1 = x_1,\ \dot{x}_2 = 0,\ \dot{x}_3 = 0$$
$$h_1(x_1, x_2, x_3) = x_2 + x_3 x_1$$
$$h_2(x_1, x_2, x_3) = x_3^2.$$

The observation algebra $H(\Sigma)$ is generated by $\mathcal{H}(\Sigma) = \{x_2 + x_3 x_1,\ x_3^2,\ x_3 x_1\}$, so

$$\operatorname{rank} \mathcal{H}'(x) = \operatorname{rank} \begin{pmatrix} x_3 & 1 & x_1 \\ 0 & 0 & 2x_3 \\ x_3 & 0 & x_1 \end{pmatrix} < 3 \Leftrightarrow$$

$\Leftrightarrow 2x_3^2 = 0$. Hence $I(X_1) = \sqrt[\mathbb{R}]{(\det \mathcal{H}')} = (x_3)$ and $X_1 = \{x : \ x = (x_1, x_2, 0)\}$. We calculate the next rank:

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}'(x) \\ g_{X_1}'(x) \end{pmatrix} = \begin{pmatrix} x_3 & 1 & x_1 \\ 0 & 0 & 2x_3 \\ x_3 & 0 & x_1 \\ 0 & 0 & 1 \end{pmatrix} < 3$$

$\Leftrightarrow x_3 = 0$. Then $X_2 = X_1$, so $I(X_2) = (x_3)$ and by Theorem 2.2 Σ is not stably locally observable. □

Example 2.2 Let Σ be the system

$$\dot{x}_1 = 0,\ \dot{x}_2 = 0,\ \dot{x}_3 = 0$$
$$h_1(x_1, x_2, x_3) = x_1 - \tfrac{1}{2} x_2^2$$
$$h_2(x_1, x_2, x_3) = x_1 x_2$$
$$h_3(x_1, x_2, x_3) = x_2^2 - x_1^3 + x_3.$$

The observation algebra $H(\Sigma)$ is generated by $\mathcal{H}(\Sigma) = \{x_1 - \frac{1}{2} x_2^2,\ x_1 x_2,\ x_2^2 - x_1^3 + x_3\}$, so

$\operatorname{rank} \mathcal{H}'(x) =$

$$\operatorname{rank} \begin{pmatrix} 1 & -x_2 & 0 \\ x_2 & x_1 & 0 \\ -3x_1^2 & 2x_2 & 1 \end{pmatrix} < 3$$

$\Leftrightarrow x_1 + x_2^2 = 0$. Hence $I(X_1) = \sqrt[\mathbb{R}]{(\det \mathcal{H}')} = (x_1 + x_2^2)$ and $X_1 = \{x : x_1 + x_2^2 = 0\}$. We calculate the next rank

$$\operatorname{rank} \begin{pmatrix} \mathcal{H}'(x) \\ g_{X_1}'(x) \end{pmatrix} =$$

$$= \operatorname{rank} \begin{pmatrix} 1 & -x_2 & 0 \\ x_2 & x_1 & 0 \\ -3x_1^2 & 2x_2 & 1 \\ 1 & 2x_2 & 0 \end{pmatrix} < 3$$

$\Leftrightarrow x_1 + x_2^2 = 0$ and $x_2 = 0$, so $I(X_2) = (x_1 + x_2^2,\ x_2)$ and

$$\operatorname{rank}_{\mathcal{O}_{X_2}} \begin{pmatrix} 1 & -x_2 & 0 \\ x_2 & x_1 & 0 \\ -3x_1^2 & 2x_2 & 1 \\ 1 & 2x_2 & 0 \\ 0 & 1 & 0 \end{pmatrix}_{/I(X_2)} = 3,$$

hence by Theorem 2.2 Σ is stably locally observable. □

3 Proof of the main result

Assume that $\mathcal{F}$ is a family of analytic functions on real analytic manifold M. For $x \in M$ let $S_x(\mathcal{F})$ be the germ at x of the level set of the family $\mathcal{F}$ (i.e. the intersection of the level sets of all $\varphi \in \mathcal{F}$ passing through x). We say that $\mathcal{F}$ is *locally observable at $x_0 \in M$* if $S_{x_0}(\mathcal{F}) = \{x_0\}$, i.e. the germ of the level set reduces to the point x_0. We say that $\mathcal{F}$ is *stably locally observable at* x_0 if there is a neighborhood U of x_0 such that $\mathcal{F}$ is locally observable at any $x \in U$.

We shall need the following lemma.

Lemma 3.1 [3] *System Σ on a manifold M, given by* (1) and (2), *is stably locally observable at $x_0 \in M$ iff the family $\mathcal{H}(\Sigma)$ is stably locally observable.* $\square$

Lemma 3.2 *Let $M = \mathbb{R}^n$,*

$$X = \{x \in \mathbb{R}^n : rank\,\mathcal{H}_0'(x) < n\}$$

and assume that the germ of X at x_0 is a germ of manilold. Denote by $\mathcal{H}(\Sigma)_{|X}$ the restriction of $\mathcal{H}(\Sigma)$ to X. Then
$\mathcal{H}(\Sigma)$ is stably locally observable at x_0 iff $\mathcal{H}(\Sigma)_{|X}$ is stably locally observable at x_0.

Proof:
" $\Longrightarrow$ " Suppose that $\mathcal{H}(\Sigma)_{|X}$ is not stably locally observable at x_0. This means that arbitrarily close to the point x_0 there exist points $x \in X$ such that through x there passes a non trivial level set of $\mathcal{H}(\Sigma)_{|X}$, denoted by $S_x(\mathcal{H}(\Sigma)_{|X})$. Then $S_x(\mathcal{H}(\Sigma)_{|X}) \subset S_x(\mathcal{H}(\Sigma))$. Therefore the family $\mathcal{H}(\Sigma)$ has also a non trivial level set near the point x_0, what means that $\mathcal{H}(\Sigma)$ is not stably locally observable at x_0.
"$\Longleftarrow$" Suppose that $\mathcal{H}(\Sigma)$ is not stably locally observable at x_0. Then the family $\mathcal{H}(\Sigma)$ has a non-trivial level set $S_x(\mathcal{H}(\Sigma)) \neq \{x\}$ for some point x in any neighborhood of x_0 and from definition of X, $x \in X$ and $S_x(\mathcal{H}(\Sigma))$ must be contained in X. Hence $\mathcal{H}(\Sigma)_{|X}$ is not stably locally observable either. $\square$

Proof of Theorem 2.2

For simplicity we consider only the case $g_{X_1} = g_1$ and $g_{X_2} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$. The other cases are similar.
Let $\mathcal{O}_{X_2} = \mathcal{O}_3/I(X_2)$, $\quad I(X_2) = (g_1, g_2)$. X_1 is a germ of manifold, hence we may write $g_1 = x_3 - \eta(x_1, x_2)$. Then we have:

$$\text{rank}_{\mathcal{O}_{X_2}} \begin{pmatrix} \mathcal{H}_0'/I(X_2) \\ g_1'/I(X_2) \\ g_2'/I(X_2) \end{pmatrix} < 3 \Leftrightarrow$$

for every $k = 1, \ldots, m$:

$$\text{rank}_{\mathcal{O}_{X_2}} \begin{pmatrix} \frac{\partial\varphi_k}{\partial x_1} & \frac{\partial\varphi_k}{\partial x_2} & \frac{\partial\varphi_k}{\partial x_3} \\ \frac{\partial\eta}{\partial x_1} & \frac{\partial\eta}{\partial x_2} & -1 \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \frac{\partial g_2}{\partial x_3} \end{pmatrix}_{/I(X_2)} < 3$$

$\Leftrightarrow$ for all $k = 1, \ldots, m$:

$$\text{rank}_{\mathcal{O}_{X_{21}}} \begin{pmatrix} \frac{\partial\varphi_{k1}}{\partial x_1} & \frac{\partial\varphi_{k1}}{\partial x_2} \\ \frac{\partial g_{21}}{\partial x_1} & \frac{\partial g_{21}}{\partial x_2} \end{pmatrix}_{/I(X_{21})} < 2 \qquad (4)$$

where $\varphi_{k1} = \varphi_k(x_1, x_2, \eta(x_1, x_2))$ and the set-germ $X_{21} \subset \mathbb{R}^2$ is defined by $g_{21}(x_1, x_2) := g_2(x_1, x_2, \eta(x_1, x_2)) = 0$.
Let $\mathcal{H}_1(x_1, x_2) := \mathcal{H}_0(x_1, x_2, \eta(x_1, x_2))$. Then the set germ $X_{12} := \{(x_1, x_2) \in \mathbb{R}^2 : \text{rank}\,\mathcal{H}_1'(x_1, x_2) < 2\}_{x_0}$ is equal to X_{21}.
Thus (4) $\Leftrightarrow$ $\text{rank}_{\mathcal{O}_{X_{12}}} \begin{pmatrix} \mathcal{H}_1'/I(X_{12}) \\ g_{12}'/I(X_{12}) \end{pmatrix} < 2$ (where $(g_{12}) = I(X_{12})$) $\Leftrightarrow$ $\mathcal{H}_1$ not stably locally observable at x_0 iff Σ not stably locally observable at x_0 (by Theorem 2.1 and Lemmas 3.1 and 3.2). $\square$

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