Algebraic criteria for stable local observability of analytic systems on $\mathbb{R}^n$

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Abstract

A new necessary and sufficient condition for stable local observability of analytic systems on $\mathbb{R}^n$ is presented. It has a form of a rank condition and can be checked in practice. A generic example is given.

Key Words. Nonlinear control, nonlinear dynamics, observers.

1 Introduction

In [5],[6] we proved a necessary and sufficient condition for stable local observability of analytic systems on $\mathbb{R}^2$ and $\mathbb{R}^3$. In the present paper we extend this condition for analytic systems on $\mathbb{R}^n$. The condition is expressed with the rank of some matrices over the rings of the germs of analytic functions. We use the germs of functions and sets, because we need only the local information about functions and sets. We assume that set-germs that appear are unions of germs of manifolds. Our condition is easy to check and may be important in practice. We give a sketch of the proof of the main theorem.

Let Σ be an analytic control system on $\mathbb{R}^n$ given by the equations

$$\dot{x} = f_0(x) + \Sigma_{i=1}^m u_i f_i(x) \quad (1)$$
$$y = h(x). \quad (2)$$

We assume that $u(t)$ is an element of an open set $\Omega \subset \mathbb{R}^m$ and we take only piecewise constant controls u; h is a C^ω map from $\mathbb{R}^n$ to $\mathbb{R}^r$.

By $x(t, x_0, u)$ we denote the solution of (1) corresponding to the initial condition $x(0, x_0, u) = x_0$ and control u, and evaluated at time t.

We say that $x_1,\ x_2\ \in \mathbb{R}^n$ are *indistinguishable* (with respect to Σ) if

$$h(x(t, x_1, u)) = h(x(t, x_2, u)), \quad (3)$$

for every control u and for every time $t \geq 0$, for which both sides of (3) are defined. Otherwise, x_1 and x_2 are *distinguishable* (with respect to Σ).

We say that Σ is *locally observable* at $x_0 \in \mathbb{R}^n$ if there is a neighborhood U of x_0 such that for every $x \in U$, x_0 and x are distinguishable.

We say that Σ is *stably locally observable* at x_0 if there is an open neighborhood V of x_0 such that for every $x \in V$, Σ is locally observable at x.
The stability of local observability means that local observability is preserved when the initial state is slightly changed or when it is not known exactly.

By the *observation algebra* of the system Σ we mean the smallest subalgebra of $C^\omega(\mathbb{R}^n, \mathbb{R})$ containing all components of h and closed under Lie derivatives with respect to vector fields of Σ. It is denoted by $H(\Sigma)$.
The observation algebra $H(\Sigma)$ is generated by the set $\mathcal{H}(\Sigma)$ consisting of functions of the form $L_{f_{i_k}} \dots L_{f_{i_1}} h_j$, where $j = 1, \dots, r;\ k \geq 0;$ $i_s = 0, \dots m$.

Let $\mathcal{O}_n$ denote the algebra over $\mathbb{R}$ of germs at x of C^ω functions on $\mathbb{R}^n$. Define m_x to be the maximal

ideal of $\mathcal{O}_n$. It consists of all the germs that vanish at x.
By I_x we mean the ideal of $\mathcal{O}_n$ generated by germs of those functions from $H(\Sigma)$ which vanish at x ($I_x \subset m_x$).
Let $\sqrt[\mathbb{R}]{I}$ denote the *real radical* of an ideal I in a commutative ring R. It is defined as the set of all elements $a \in R$ such that there are integers $m > 0$, $k \geq 0$ and $b_1, b_2, \ldots, b_k \in R$ such that $a^{2m} + b_1^2 + \cdots + b_k^2 \in I$.

A necessary and sufficient condition of local observability was proved by Z. Bartosiewicz in [1].

Theorem 1.1 *The following conditions are equivalent:*
(i) $\sqrt[\mathbb{R}]{I_x} = m_x$*,*
(ii) Σ *is locally observable at* x*.* □

Now we are interested in conditions of stable local observability. We know from [2] that Hermann-Krener condition is stable.

Let $dH = \{d\varphi : \varphi \in H\}$.

Theorem 1.2 *If* $\dim dH(\Sigma, x_0) = n$ *then local observability at* x_0 *is stable.* □

A necessary and sufficient condition of stable local observability of analytic system in geometric form was given by Z. Bartosiewicz in [3].

Stable local observability of control system Σ may be expressed by stable local observability of family of analytic functions, defined below.

Assume that $\mathcal{F}$ is a family of analytic functions on real analytic manifold M. For $x \in M$ let $S_x(\mathcal{F})$ be the germ at x of the level set of the family $\mathcal{F}$ (i.e. the intersection of the level sets of all $\varphi \in \mathcal{F}$ passing through x).

We say that $\mathcal{F}$ is *locally observable* at $x_0 \in M$ if $S_{x_0}(\mathcal{F}) = x_0$, i.e. the germ of the level set reduces to the point x_0. We say that $\mathcal{F}$ is *stably locally observable* at x_0 if there is a neighborhood U of x_0 such that $\mathcal{F}$ is locally observable at any $x \in U$.

Lemma 1.1 *[3] System* Σ *on manifold* M *is stably locally observable at* $x_0 \in M$ *iff the family* $\mathcal{H}(\Sigma)$ *is stably locally observable.* □

In the next section we give an algebraic condition of stable local observability which should be easier to check.

2 Main result

Let Σ be an analytic control system on $\mathbb{R}^n$, and $x_0 \in U(x_0) \subset \mathbb{R}^n$.
We define a set

$$\tilde{X}_1 := \{x \in U(x_0) : \text{ rank } \mathcal{H}'(x) < n \},$$

where by rank $\mathcal{H}'(x)$ we mean

$$\text{rank}\mathcal{H}'(x) = \max_{s \geq 1} \text{rank} \begin{pmatrix} \varphi_1'(x) \\ \vdots \\ \varphi_s'(x) \end{pmatrix},$$

where $\varphi_i \in \mathcal{H}, i = 1, \ldots, s$.
$\tilde{X}_1$ is an analytic set. We shall assume that the germ of $\tilde{X}_1$ at x_0, denoted by X_1, is a union of germs of analytic manifolds.

Lemma 2.1 *Let* $\mathcal{H}(\Sigma)_{|\tilde{X}_1}$ *denote the restriction of* $\mathcal{H}(\Sigma)$ *to* $\tilde{X}_1$*. Then*
$\mathcal{H}(\Sigma)$ *is stably locally observable at* x_0 *iff* $\mathcal{H}(\Sigma)_{|\tilde{X}_1}$ *is stably locally observable at* x_0*.* □

Now we will construct a finite sequence of set-germs X_k and a corresponding sequence of ideals I_k. By $\tilde{X}_k$ we will denote a representative of X_k.
Let $X_0 := U_{x_0}$ be the germ of a neighborhood of x_0 and $I_0 = \{0\}$ is a null ideal (this is an ideal of all the function-germs vanishing on X_0)
Now let

$$X_1 := \{x \in \tilde{X}_0 : \text{ rank } \mathcal{H}'(x) < n \}_{x_0},$$

where the subscript x_0 means the germ at x_0.
Let $I_1 := I(X_1)$ be the ideal of function-germs on $\mathbb{R}^n$ at x_0 vanishing on X_1.

Assume that we constructed a set-germ X_k and X_k is a union of germ of manifold.
Let $I_k := I(X_k)$. Then $I_k = (g_1, \ldots, g_p)$ and $g_i \in \mathcal{O}_n$.
We choose the generators of I_k in such a way that

$g_i \notin (g_1, \dots, g_{i-1}), i = 1, \dots, p.$
Now let

$$X_{k+1} := \{x \in \tilde{X}_k : \text{rank} \begin{pmatrix} \mathcal{H}'(x) \\ g'_{X_k}(x) \end{pmatrix} < n\}_{x_0},$$

where $g'_{X_k}(x) = \begin{pmatrix} \tilde{g}'_1(x) \\ \vdots \\ \tilde{g}'_p(x) \end{pmatrix}$ and $\tilde{g}_1, \dots, \tilde{g}_p$ are the representatives of generators of the ideal I_k. We define also $I_{k+1} := I(X_{k+1})$.

The following proposition allows to compute the ideals I_k in many cases.

Proposition 2.1

$$I_{k+1} = \sqrt[\mathbb{R}]{V(\mathcal{H}'_0, g'_1, \dots, g'_p)},$$

where $V(\mathcal{H}'_0, g'_1, \dots, g'_p)$ denote the ideal in $\mathcal{O}_n$ generated by all determinants of the form

$$\begin{vmatrix} \varphi'_1 \\ \vdots \\ \varphi'_n \end{vmatrix}, \ \varphi_i \in \{\mathcal{H}_0, g_1, \dots, g_p\}, i = 1, \dots, n.$$

($\mathcal{H}_0$ is a set of germs at x_0 of functions from $\mathcal{H}$). □

Observe that

$$I_1 = \sqrt[\mathbb{R}]{V(\mathcal{H}'_0)}.$$

Remark 2.1 *Let $I_k = (g_1, \dots, g_p)$, then*

1. *If for all $x \in \tilde{X}_k$ rank $\begin{pmatrix} \mathcal{H}'(x) \\ g'_{X_k}(x) \end{pmatrix} < n$ then $X_{k+1} = X_k$ and $I_{k+1} = I_k = (g_1, \dots, g_p)$.*
2. *If $X_{k+1} \subset X_k$ then $I_{k+1} \supset I_k$.*
3. *For all $x \in \tilde{X}_k$ rank $\begin{pmatrix} \mathcal{H}'(x) \\ g'_{X_k}(x) \end{pmatrix} = n$ iff $X_{k+1} = \emptyset$. (In this situation $I_{k+1} = \mathcal{O}_n$).* □

We assume that for all k, X_k are unions of the germs of manifolds. They form a decreasing finite sequence of set-germs

$$X_0 \supset X_1 \supset \dots \supset X_k \supset \dots \supset X_r. \tag{4}$$

Because the manifolds are finite-dimensional, this sequence stabilizes for some r.

Lemma 2.2

$$r \leq n + 1.$$

□

To the sequence (4) there corresponds an increasing sequence of ideals

$$I_0 \subset I_1 \subset I_2 \subset \dots \subset I_k \subset \dots \subset I_r.$$

Now we can state the main result of the paper.

Theorem 2.1 *An analytic system Σ is stably locally observable at $x_0 \in \mathbb{R}^n$ iff there exists $1 \leq k \leq n+1$ such that*

$$I_k = \mathcal{O}_n.$$

□

Remark 2.2 *The condition in Theorem 2.1 means that $X_k = \emptyset$.*

Example 2.1 Let Σ be the sysytem

$$\begin{aligned} \dot{x}_1 &= x_1 u_1 \\ \dot{x}_2 &= x_2 u_2 \\ &\vdots \\ \dot{x}_{n-1} &= x_{n-1} u_{n-1} \\ \dot{x}_n &= 0 \\ y &= x_1^2 + x_2^2 + \dots + x_n^2 \end{aligned}$$

Then $\mathcal{H}(\Sigma) = \{x_1^2, x_2^2, \dots, x_{n-1}^2, x_n^2\}$ and

$$\text{rank}\, \mathcal{H}'(x) = \text{rank} \begin{pmatrix} 2x_1 & 0 & \dots & 0 \\ 0 & 2x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2x_n \end{pmatrix} < n \Leftrightarrow$$

$$\Leftrightarrow x_1 x_2 \dots x_n = 0.$$

Hence $I_1 = (x_1 x_2 \dots x_n)$ and X_1 is union of set-germs. Now

$$\text{rank} \begin{pmatrix} \mathcal{H}'(x) \\ g'_{X_1}(x) \end{pmatrix} < n \Leftrightarrow$$

$$\text{rank} \begin{pmatrix} 2x_1 & 0 & \dots & 0 \\ 0 & 2x_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 2x_n \\ x_2 \dots x_n & x_1 x_3 \dots x_n & \dots & x_1 x_2 \dots x_{n-1} \end{pmatrix} < n$$

$$\Leftrightarrow x_1 x_2 \ldots x_{n-1} = 0 \wedge \ldots \wedge x_2 \ldots x_n = 0.$$

Hence

$I_2 = (x_1 \ldots x_{n-1}, x_1 \ldots x_{n-2} x_n = 0, \ldots x_2 \ldots x_n = 0).$

I_2 has n generators, they are products of n coordinates functions. The next step is to calculate

$$\mathrm{rank} \begin{pmatrix} \mathcal{H}'(x) \\ g'_{X_2}(x) \end{pmatrix} < n \Leftrightarrow$$

$$\Leftrightarrow x_1 x_3 \ldots x_{n-1} = 0 \wedge \ldots \wedge x_3 x_4 \ldots x_n = 0.$$

I_3 is generated by all products of $(n-2)$ coordinates functions. And I_n is generated by $(x_1, x_2, \ldots, x_n)$ and so $\mathrm{rank} \begin{pmatrix} \mathcal{H}'(x) \\ g'_{X_n}(x) \end{pmatrix} = n$ for all x.

Hence $X_{k+1} = \emptyset$ and $I_{k+1} = \mathcal{O}_n$. Therefore from Theorem 2.1 Σ is stably locally observable at every point. □

3 Sketch of the proof of the main result

We shall sketch the proof of Theorem 2.1 assuming for simplicity that X_k are actually germs of analytic manifolds.

Let $I(X_k) = (g_1, \ldots, g_p)$. Hence every point $x \in \tilde{X}_k$ satisfies equations

$$\tilde{g}_1(x) = \ldots = \tilde{g}_p(x) = 0.$$

Moreover X_k is a germ of manifold, hence we may write

$$\begin{aligned} \tilde{g}_1(x) &= x_1 - \eta_1(x_{p+1}, \ldots, x_n) \\ &\vdots \\ \tilde{g}_p(x) &= x_p - \eta_p(x_{p+1}, \ldots, x_n). \end{aligned} \tag{5}$$

We may assume that $\tilde{X}_k$ is an analytic manifold. Then for some open set $G \subset \mathbb{R}^{n-p}$ the map

$$\Phi : G \to \tilde{X}_k, \tag{6}$$

$\Phi(\tilde{x}) = (\eta_1(\tilde{x}), \ldots, \eta_p(\tilde{x}), \tilde{x}),\ \tilde{x} = (x_{p+1}, \ldots, x_n) \in G$ is an analytic diffeomorphism. We define $\mathcal{H}_k$ as the family of functions of the form

$$\psi(\tilde{x}) := (\varphi \circ \Phi)(\tilde{x}) = \varphi(\eta_1(\tilde{x}), \ldots, \eta_p(\tilde{x}), \tilde{x}), \tag{7}$$

where $\varphi \in \mathcal{H}$. We need the following lemmas.

Lemma 3.1 *The restriction of family $\mathcal{H}$ to $\tilde{X}_k$ is stably locally observable at $x_0 \in \tilde{X}_k$ iff $\mathcal{H}_k$ is stably locally observable at $\tilde{x}_0 = (x_{0,p+1}, \ldots, x_{0,n})$.* □

Lemma 3.2

$$\forall\, (0 \le i \le n+1)\ X_i \supset S_x(\mathcal{H}).\Box \tag{8}$$

Proof of Theorem 2.1
(inductively with regard to n).
1) n=1
Hence the statement is follows
$\mathcal{H}(\Sigma)$ is stably locally observable at $x_0 \in \mathbb{R}$ iff

$$I_1 = \mathcal{O}_1 \text{ or } I_2 = \mathcal{O}_1.$$

The proof is not difficult.
2) Induction Assumption
Theorem is satisfied for all $\mathcal{H}(\Sigma)$ on $\mathbb{R}^k,\ k < n$.
3) We must show that the following conditions are equivalent:
(a) $\mathcal{H}$ is stably locally bservable at $x_0 \in \mathbb{R}^n$,
(b) $\exists\ (1 \le\ k\ \le n+1)\ I_k = \mathcal{O}_n$.

(b) $\Longrightarrow$(a) Assume that (a) is not satisfied. The condition (b) means that there exists $k \le n+1$ such that $X_k = \emptyset$. But if $\mathcal{H}$ is not stably locally observable at x_0, then arbitrarily near x_0 there exists a point x such that $S_x(\mathcal{H}) \neq \{x\}$. And from Lemma 3.2
dim $X_i > 0$, what is in contradiction with condition (b). □

(a) $\Longrightarrow$ (b) Assume that (b) is not satisfied. Consider k such that $I_{k-1} \subset I_k = I_{k+1} \neq \mathcal{O}_n$. Let $I_k = (g_1, \ldots, g_p)$, $k \le p < n$, then dim $\tilde{X}_k = n - p$. Moreover $X_{k+1} = X_k$, hence

$$\forall x \in \tilde{X}_k\ \mathrm{rank} \begin{pmatrix} \mathcal{H}'(x) \\ \tilde{g}'_1(x) \\ \vdots \\ \tilde{g}'_p(x) \end{pmatrix} < n. \tag{9}$$

One can show that (9) is equivalent to

$$\forall \tilde{x} \in G\ \mathrm{rank} \begin{pmatrix} \frac{\partial \psi_1}{\partial x_{p+j}}(\tilde{x}) \\ \vdots \\ \frac{\partial \psi_{n-p}}{\partial x_{p+j}}(\tilde{x}) \end{pmatrix}_{j=1,\ldots,n-p} < n - p, \tag{10}$$

where from (7)

$$\frac{\partial \psi_i}{\partial x_{p+j}}(\tilde{x}) = \frac{\partial \varphi_i}{\partial x_{p+1}}(\Phi(\tilde{x})) + \sum_{i=1}^{p} \frac{\partial \varphi_i}{\partial x_i}(\Phi(\tilde{x})) \frac{\partial \eta_i}{\partial x_{p+j}}(\Phi(\tilde{x}))$$

and Φ was defined in (6) as locally a diffeomorphism. The statement (10) is equivalent to

$$\forall \tilde{x} \in G \ \ \mathrm{rank}\mathcal{H}'_k(\tilde{x}) < n - p, \tag{11}$$

Now we repeat the constructions of set-germs for the family functions $\mathcal{H}_k$ and define

$$Y_1 := \{\tilde{x} \in G : \mathrm{rank}\mathcal{H}'_k(\tilde{x}) < n - p\}_{\tilde{x}_0}$$

From (11) we have that we may choose the representative of germ Y_1 :
$\tilde{Y}_1 = G$ and for family $\mathcal{H}_k$ defined on $\mathbb{R}^{n-p}$,
$I(Y_1) = \{0\}$. Therefore the ideals stabilize in the beginning and $I(Y_1) \neq \mathcal{O}_{n-p}$. Hence from the inductive assumption $\mathcal{H}_k$ is not stably locally observable at $\tilde{x}_0 \in \mathbb{R}^{n-p}$ and from Lemma 3.1 $\mathcal{H}_{|\tilde{X}_k}$ is not stably locally observable at $x_0 \in \tilde{X}_k \subset \mathbb{R}^n$. Hence from Lemma 2.1 $\mathcal{H}$ is not stably locally observable at x_0 .
$\square$

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