Realizations of linear control systems on time scales*

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Abstract

Linear constant-coefficients control systems with output on arbitrary time scales are studied. Kalman criteria of controllability and observability are extended to such systems. The main problem is to find criteria for an abstract input/output map to have a realization as a system on the time scale. Two different characterizations of realizability are proved. They extend the classical results obtained for continuous-time and discrete-time systems. Minimal realizations and their uniqueness are also studied.

1 Introduction

The problem of realization of an input-output map means finding a dynamical state-space system with input and output, able to reproduce, when initialized at some state, the given input/output behavior. In the linear case the results obtained for continuous-time and discrete-time are very similar. A natural question is whether these two cases may be unified in one more general theory.

Calculus on time scales, originated in 1988 by Stefan Hilger [6], seems to deliver a perfect language for unification of continuous-time and discrete-time theories. A time scale is a model of time. It is an arbitrary closed subset of the real line. Besides the standard cases of the whole line (continuous time) and the set of integers (discrete time), there are numerous examples of other time models, which may be partly continuous and partly discrete. The delta derivative of a real function defined on the time scale is a generalization of the classical (time) derivative for continuous time and the finite forward difference for the discrete time. Similarly, the integral of a real function defined on the time scale is an extension of the Riemann integral in the continuous time and the finite sum in the discrete time. As a consequence, differential equations

*The work supported by the Białystok Technical University grants No. W/IMF/1/04 and W/IMF/3/06

as well as difference equations are naturally accommodated in this theory. We recall the basic facts on systems of linear delta differential equations. However we drop the assumption about regressivity of the matrix of coefficients of the system. Regressivity implies existence and uniqueness of forward and backward solutions of the system of linear delta differential equations (see [3]). In the problems we study here only forward solutions are needed and they exist without the regressivity assumption.

In this paper we consider linear constant-coefficients control systems with output, defined on time scales. The state space model described by delta differential equations and output equations gives rise to the input/output map with an integral operator on the time scale. The standard problem is now to construct a state-space representation of an abstract input/output map and to give conditions for this map that allow for such a representation. The main result (Theorem 5.2) may be seen as an extension of the classical criterion (see e.g. [4]). However one has to take into account that the input/output map for a system on a time scale is not in general a convolution operator in the standard sense (either continuous or discrete). This makes the theory more complicated. It is much easier to construct a realization with time-varying matrices. This was done by L.V. Fausett and K.N. Murty [5], who studied systems with coefficients depending on time.

In the continuous-time case the input/output map (or the transfer matrix) is often represented by the Markov parameters, which may be defined as the derivatives at 0 of the kernel matrix of the convolution operator that defines the input/output map. We introduce the Markov parameters for input/output maps on time scales and show that the input/output map has a state-space realization if and only if the Markov parameters satisfy a linear recurrence equation. However a theory of power series on time scales is missing, so we have to assume something that allows to recover a function from the sequence of its derivatives at some point (we call this analyticity).

We also show that, like in the theory of continuous-time systems, one can always reduce the dimension of the state space of the constructed realization so that the reduced system is controllable and observable (i.e. minimal). For completeness we add a section devoted to these properties, showing that the standard Kalman conditions (see e.g. [7, 9]) are still valid for systems on time scales. These results were earlier announced in [1] and, partly, in [5]. But [1] contained only sketches of the proofs. On the other hand, the proof of the Kalman condition for controllability in [5] (Theorem 3.4) is correct only for the time scale equal to $\mathbb{R}$ (i.e. in the classical continuous-time case), as the standard continuous-time exponential matrix and its properties were used in the calculations. In the general case the proof is more complicated and we give it here.

The first attempt to unify the theories of continuous-time and discrete-time control systems was undertaken by R.H. Middleton and G.C. Goodwin in [8]. They studied controllability, observability and realizations of linear systems.

2 Calculus on time scales

For convenience of the reader we give here a short introduction to differential calculus on time scales. This is a generalization of the standard differential calculus, on one hand, and the calculus of finite differences, on the other hand. Then we describe the inverse operation — integration. This will allow to solve differential equations on time scales. More material on this subject can be found in [3].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. If $t_0, t_1 \in \mathbb{T}$, then $[t_0, t_1]$ denotes the intersection of the ordinary closed interval with $\mathbb{T}$. Similar notation is used for open, half-open or infinite intervals. For $t \in \mathbb{T}$ we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ if $t \neq \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ if $t \neq \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\sigma(t) < t$, it is called *left-scattered*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\sigma(t) = t$, then t is *left-dense*.

We define also the set $\mathbb{T}^k := \{t \in \mathbb{T} : t \text{ nonmaximal or left-dense}\}$. Thus $\mathbb{T}^k$ is got from $\mathbb{T}$ by removing its maximal point if this point exists and is left-scattered. The delta derivative, introduced in Definition 2.2, will be well defined only on $\mathbb{T}^k$. Finally, we denote $f^\sigma := f \circ \sigma$ for any function $f : \mathbb{T} \to \mathbb{R}$.

Example 2.1. **a)** If $\mathbb{T} = \mathbb{R}$ then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
b) If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.
c) Let $q > 1$. We define time scale $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$. Then $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$ and $\mu(t) = (q-1)t$ for all $t \in \mathbb{T}$.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. *The delta derivative* of f at t, denoted by $f^\triangle(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\triangle(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^k$.

Remark 2.3. In general, the function σ may not be delta differentiable.

Remark 2.4. **i)** If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff $f^{\triangle}(t) = \lim\limits_{s \to t} \frac{f(t)-f(s)}{t-s} = f'(t)$ exists, i.e. iff f is differentiable in the ordinary sense at t.
ii) If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^{\triangle}(t) = \frac{f(\sigma(t))-f(t)}{\mu(t)} = f(t+1) - f(t)$.
iii) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, then $f^{\triangle}(t) = \frac{f(qt)-f(t)}{(q-1)t}$ for all $t \in \mathbb{T}$, $t \neq 0$ and for all functions $f : \mathbb{T} \to \mathbb{R}$.

Delta derivatives of higher order are defined in the standard way: $f^{\triangle^k}$ will denote the delta derivative of order k. We often drop the word 'delta' and talk only about derivatives and differentiability.

Example 2.5. The delta derivative of t^2 is $t+\sigma(t)$. This means that the second delta derivative of t^2 may not exist.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points at $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. It can be shown that

- f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated
- σ is rd-continuous.

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. It can be proved that if f is regulated then there exists a function F that is pre-differentiable with region of differentiation D such that $F^{\triangle}(t) = f(t)$ for all $t \in D$. Any such function is called a pre-antiderivative of f. Then the *indefinite integral* of f is defined by $\int f(t) \triangle t := F(t) + C$ where C is an arbitrary constant. *Cauchy integral* is defined by

$$\int_r^s f(t) \triangle t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}^k$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

Remark 2.6. It can be shown that every rd-continuous function has an antiderivative. Moreover, if $f(t) \geq 0$ for all $a \leq t \leq b$ and $\int_a^b f(\tau) \triangle \tau = 0$, then $f \equiv 0$.

Example 2.7. **a)** If $\mathbb{T} = \mathbb{R}$, then $\int\limits_a^b f(\tau)\triangle\tau = \int\limits_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral.
b) If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int\limits_a^b f(\tau)\triangle\tau = \sum\limits_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

Remark 2.8. An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula of an antiderivative of t: the derivative of $\frac{t^2}{2}$ is $\frac{t+\sigma(t)}{2} = t + \frac{\mu(t)}{2}$.

3 Exponential function. Linear systems on time scale

Let us consider the system of delta differential equations on time scale $\mathbb{T}$:

$$x^{\triangle}(t) = Ax(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and A is a constant $n \times n$ matrix.

Proposition 3.1. *Equation (1) with initial condition $x(t_0) = x_0$ has a unique forward solution defined for all $t \in [t_0, +\infty)$.*

Proof. See [3]. □

The *matrix exponential function* (at t_0) for A is defined as the unique forward solution of the matrix differential equation $X^{\triangle} = AX$, with the initial condition $X(t_0) = I$. Its value at t is denoted by $e_A(t, t_0)$.
Remark 3.2. When the matrix A is regressive, i.e. $I + \mu(t)A$ is invertible for any $t \in \mathbb{T}^k$, the solution of (1) is unique and global [3]. Roughly speaking, regressivity of the matrix A means that we can solve equation (1) both backward and forward. In control theory we need usually only forward solutions, so we do not assume this property.

Example 3.3. **a)** If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$.
b) If $\mathbb{T} = \mathbb{Z}$, then $e_A(t, t_0) = (I + A)^{(t-t_0)}$.
c) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$, $q > 1$, then $e_A(q^k t_0, t_0) = \prod_{i=0}^{k-1}(I + (q-1)q^i t_0 A)$ for $k \geq 1$ and $t_0 > 0$.

Let us consider now a nonhomogeneous system

$$x^{\triangle}(t) = Ax(t) + f(t) \tag{2}$$

where f is rd-continuous.

Theorem 3.4. *Let $t_0 \in \mathbb{T}$. System (2) for the initial condition $x(t_0) = x_0$ has a unique forward solution of the form*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\triangle\tau. \tag{3}$$

Proof. For a regressive matrix A the theorem was shown in [3]. In the general case the proof is the same, but we can claim only existence of a forward solution. □

Theorem 3.5. (Putzer Algorithm) *Let A be an $n \times n$ matrix and $t_0 \in \mathbb{T}$. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then for $t \geq t_0$*

$$e_A(t, t_0) = \sum_{i=0}^{n-1} r_{i+1}(t, t_0) P_i$$

where $r(t, t_0) := (r_1(t, t_0), \ldots, r_n(t, t_0))^T$ is the forward solution (evaluated at t) of the initial value problem

$$r^{\triangle} = \begin{pmatrix} \lambda_1 & 0 & 0 & \ldots & 0 \\ 1 & \lambda_2 & 0 & \ldots & 0 \\ 0 & 1 & \lambda_3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \vdots & 1 & \lambda_n \end{pmatrix} r, \quad r(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the matrices $P_0, P_1, \ldots, P_n$ are recursively defined by $P_0 = I$ and

$$P_{k+1} = (A - \lambda_{k+1} I) P_k, \quad \text{for } k = 0, 1, \ldots, n-1$$

Proof. It is similar to the one given under regressivity assumption for matrix A in [3]. □

Proposition 3.6. *The following properties hold for every $t, s, r \in \mathbb{T}$ such that $r \leq s \leq t$:*
i) *$e_A(t, t) = I$;*
ii) *$e_A(t, s) = e_A^{-1}(s, t)$ under the regressivity condition;*
iii) *$e_A(t, s) e_A(s, r) = e_A(t, r)$;*
iv) *$A e_A(t, s) = e_A(t, s) A$.*

Proof. Property i) follows from the definition. Property ii) was shown in [3]. Property iii) follows from the uniqueness of forward solutions of delta differential equations.
iv) Fix $s \in \mathbb{T}$ and consider function $F(t) = A e_A(t, s) - e_A(t, s) A$ for $t \geq s$. Then $F(s) = 0$. Moreover $F^{\triangle}(t) = A^2 e_A(t, s) - A e_A(t, s) A = A(A e_A(t, s) - e_A(t, s) A) = A F(t)$. F solves delta differential equation $F^{\triangle}(t) = A F(t)$, $F(s) = 0$. By Proposition 3.1 we have $F \equiv 0$, which gives the result. □

Let $e_A^{t_0}(t) = e_A(t, t_0)$. It is easy to see that $e_A^{t_0}$ is infinitely many times differentiable at t_0 and $(e_A^{t_0})^{\triangle^k}(t_0) = A^k$.

Let ${}_{t_0}e_A(s) = e_A(t_0, s)$ for $s \leq t_0$. Then $e_A(t, s) = e_A^{t_0}(t) \cdot {}_{t_0}e_A(s)$.

Proposition 3.7. *$({}_{t_0}e_A)^{\triangle}(s) = -({}_{t_0}e_A \circ \sigma)(s) A$ for $s \leq \rho(t_0)$.*

Proof. Let $\mu(s) > 0$ and $s < t_0$. Then, from the definition of delta derivative, it follows

$$
\begin{aligned}
({}_{t_0}e_A)^{\triangle}(s) &= \frac{1}{\mu(s)}[e_A(t_0, \sigma(s)) - e_A(t_0, s)] \\
&= \frac{1}{\mu(s)}[e_A(t_0, \sigma(s)) - e_A(t_0, \sigma(s))e_A(\sigma(s), s)] \\
&= -e_A(t_0, \sigma(s))\frac{e_A(\sigma(s), s) - e_A(s, s)}{\mu(s)} = -e_A(t_0, \sigma(s))A
\end{aligned}
$$

Now, let $\mu(s) = 0$ and $\triangle s > 0$ be such that $s + \triangle s < t_0$. Then we have

$$
\begin{aligned}
({}_{t_0}e_A)^{\triangle}(s) &= \lim_{\Delta s \to 0} \frac{e_A(t_0, s + \Delta s) - e_A(t_0, s)}{\Delta s} \\
&= \lim_{\Delta s \to 0} \frac{e_A(t_0, s + \Delta s) - e_A(t_0, s + \Delta s)e_A(s + \Delta s, s)}{\Delta s} \\
&= \lim_{\Delta s \to 0} \frac{-e_A(t_0, s + \Delta s)[e_A(s + \Delta s, s) - e_A(s, s)]}{\Delta s} \\
&= -e_A(t_0, \sigma(s))A.
\end{aligned}
$$

Similarly for $\triangle s < 0$. Now we also admit the case $s = t_0$ if $\rho(t_0) = t_0$. □

4 Controllability and observability

Let $n \in \mathbb{N}$ be fixed. In this section we shall assume that the time scale $\mathbb{T}$ consists of at least $n + 1$ elements.

Let us consider a linear control system with output, denoted by Σ, and defined on the time scale $\mathbb{T}$:

$$x^{\triangle}(t) = Ax(t) + Bu(t) \tag{4a}$$

$$y(t) = Cx(t) \tag{4b}$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$. Recall that:

- System Σ is *controllable* if for any two states $x_0, x_1 \in \mathbb{R}^n$ there exist $t_0, t_1 \in \mathbb{T}$, $t_1 > t_0$, and a piecewise rd-continuous control $u(t)$, $t \in [t_0, t_1]$, such that for $x_0 = x(t_0)$ one has $x(t_1) = x_1$.
- Two states $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* if for every control u and for every time $t \in \text{dom}u = [t_0, t_u]$ the value of the output $y(t)$ corresponding to u is the same for both initial conditions $x(t_0) = x_1$ and $x(t_0) = x_2$. System Σ is *observable* if any two indistinguishable states are equal.
- System Σ is *minimal* if it is both controllable and observable.

Let us consider the set of all points that can be reached at time t_1 starting from x_0 at time t_0 (*reachability set*). It will be denoted by $\mathcal{R}_{x_0}(t_0, t_1)$. Observe that for $x_0 = 0$ this set is a linear subspace of the state space $\mathbb{R}^n$.

Theorem 4.1. *Assume that the interval* $[t_0, t_1]$ *consists of at least* $n+1$ *elements. The following conditions are equivalent:*
i) $\mathcal{R}_0(t_0, t_1) = \mathbb{R}^n$
ii) $\operatorname{rank}(P_0B, P_1B, \ldots, P_{n-1}B) = n$
iii) $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n$.

Proof. *i)* $\Rightarrow$ *ii)* Let us assume that $\operatorname{rank}(P_0B, P_1B, \ldots, P_{n-1}B) < n$. This implies that the rows of the matrix $(P_0B, P_1B, \ldots, P_{n-1}B)$ are linearly dependent and there exists a constant vector $v \neq 0$ such that

$$v^T P_0 B = v^T P_1 B = \ldots = v^T P_{n-1} B = 0.$$

From the Putzer's algorithm we obtain that

$$x_1 = \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) B u(\tau) \triangle\tau = \int_{t_0}^{t_1} \sum_{i=0}^{n-1} r_{i+1}(t_1, \sigma(\tau)) P_i B u(\tau) \triangle\tau.$$

Then

$$v^T x_1 = \sum_{i=0}^{n-1} \int_{t}^{t_1} r_{i+1}(t_1, \sigma(\tau)) v^T P_i B u(\tau) \triangle\tau = 0$$

for all controls. So, every state x_1 reachable from 0 lies in the hyperplane of $\mathbb{R}^n$ orthogonal to v. Hence, the reachable set $\mathcal{R}_0(t_0, t_1)$ is not equal to $\mathbb{R}^n$.

ii) $\Rightarrow$ *iii)* Let $\operatorname{rank}(P_0B, P_1B, \ldots, P_{n-1}B) = n$ and let us assume that $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) < n$. This assumption implies existence of a nonzero vector v such that $v^T A^i B = 0$ for $i = 0, 1, \ldots, n-1$. From the definition of the matrices P_i it follows that

$$P_k = (A - \lambda_k I)(A - \lambda_{k-1} I) \ldots (A - \lambda_1 I) \tag{5}$$

or in the polynomial form

$$P_k = A^k + a_{k-1}^k A^{k-1} + \ldots + a_1^k A + a_0^k I = \sum_{i=0}^{k} a_i^k A^i \tag{6}$$

where a_i^k are coefficients defined by (5). Then $v^T P_k B = \sum_{i=0}^{k} a_i^k v^T A^i B = 0$ for $k = 0, 1, \ldots, n-1$. This means that matrix $(P_0B, P_1B, \ldots, P_{n-1}B)$ has dependent rows. Hence contradiction.

iii) $\Rightarrow$ *i)* Let us assume that $\mathcal{R}_0(t_0, t_1) \neq \mathbb{R}^n$. Then there exists a vector $v \neq 0$ such that for all controls u

$$v^T \int_{t_0}^{t_1} e_A(t_1, \sigma(\tau)) B u(\tau) \triangle\tau = 0 \tag{7}$$

Let us take

$$u(\tau) = B^T e_A(t_1, \sigma(\tau))^T v. \tag{8}$$

This gives

$$\int_{t_0}^{t_1} v^T e_A(t_1, \sigma(\tau)) B B^T e_A(t_1, \sigma(\tau))^T v \triangle\tau = 0$$

which implies $v^T e_A(t_1, \sigma(\tau))B = 0$ for all $\tau \in [t_0, t_1)$. Continuity of ${}_{t_1}e_A$ and density of $\sigma([t_0, t_1])$ in the interval $[\sigma(t_0), \sigma(t_1)]$ implies that $v^T e_A(t_1, \tau)B = 0$ for all $\tau \in [\sigma(t_0), t_1]$. Thus in particular $v^T B = 0$. The left hand side of this equation is delta differentiable with respect to τ (Proposition 3.7). Subsequent derivatives and the density argument as above give

$$v^T e_A(t_1, \tau)AB = 0 \text{ for } \tau \in [\sigma^2(t_0), t_1]$$
$$\vdots$$
$$v^T e_A(t_1, \tau)A^{n-1}B = 0 \text{ for } \tau \in [\sigma^n(t_0), t_1]$$

For $\tau = t_1$ we have $v^T A^k B = 0$, $k = 0, 1, \ldots, n-1$, so $\operatorname{rank}[B, AB, \ldots, A^{n-1}B] < n$. □

From the proof of Theorem 4.1 we get

Corollary 4.2. *Let the interval $[t_0, t_1]$ consist of at least $n+1$ elements. Then $\mathcal{R}_0(t_0, t_1)$ is spanned by the columns of the matrix $(B, AB, \ldots, A^{n-1}B)$.*

Corollary 4.3. *Let $x_0 \in \mathbb{R}^n$ and the interval $[t_0, t_1]$ consist of at least $n+1$ elements. Then $\mathcal{R}_{x_0}(t_0, t_1) = \mathbb{R}^n \Leftrightarrow \operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n$.*

Proof. This follows from the fact that $\mathcal{R}_{x_0}(t_0, t_1) = e_A(t_1, t_0)x_0 + \mathcal{R}_0(t_0, t_1)$ and Theorem 4.1. □

Corollary 4.4. *The system Σ is controllable if and only if $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n$.*

Proof. If the condition $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n$ is satisfied, then, by Corollary 4.3, any state x_1 can be reached from any other state x_0 on any time interval $[t_0, t_1]$ consisting of at least $n+1$ elements. On the other, if the condition $\operatorname{rank}(B, AB, \ldots, A^{n-1}B) = n$ is not satisfied, then, by Corollary 4.2, for any $t_0, t_1 \in \mathbb{T}$, $t_0 < t_1$, $\mathcal{R}_0(t_0, t_1)$ is contained in the proper subspace of $\mathbb{R}^n$ spanned by the columns of $(B, AB, \ldots, A^{n-1}B)$. Thus there are states that are not reachable from $x_0 = 0$ whatever t_0 and t_1 are chosen. □

Observability is characterized in a similar way.

Theorem 4.5. *The following conditions are equivalent:*
i) the system Σ is observable
ii) $\operatorname{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n$

Proof. Without loss of generality we may assume that $B = 0$ (controls do not influence the indistinguishability relation). Then we have $y(t) = Ce_A(t, t_0)x_0$. Assume that ii) is satisfied and let $y(t) = 0$ for $t \geq t_0$. Then also $y^{\triangle^k}(t) = 0$ for $t \geq t_0$, so $y^{\triangle^k}(t) = CA^k e_A(t, t_0)x_0 = 0$ and $y^{\triangle^k}(t_0) = CA^k x_0 = 0$ for $k \geq 0$. Then

$$\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} x_0 = 0. \tag{9}$$

Hence $x_0 = 0$. This gives observability of Σ.

Now, let us assume, that $\operatorname{rank} \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} < n$. Then there exists a nonzero vector $v \in \mathbb{R}^n$ such that $\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} v = 0$. This implies that $\begin{pmatrix} CP_0 \\ CP_1 \\ \vdots \\ CP_{n-1} \end{pmatrix} v = 0$ or, in the equivalent form, $CP_0 v = 0, \ldots, CP_{n-1}v = 0$. The Putzer algorithm implies that $Ce_A(t, t_0)v = 0$ for $t \geq t_0$. So, for $x_0 = v$ we have $y(t) = 0$ for $t \geq t_0$. This means that Σ is not observable. □

5 Markov parameters and state-space representation

Let us fix $t_0 \in \mathbb{T}$. In this section we shall assume that the set of those $t \in \mathbb{T}$ that are greater than t_0 is infinite. This assumption allows for computing (delta) derivatives at t_0 of arbitrary order.

Consider system Σ given by (2) and assume that $x(t_0) = 0$. Then for any piecewise rd-continuous input $u : [t_0, \infty) \to \mathbb{R}^m$ and any $t \in [t_0, \infty)$

$$y(t) = \int_{t_0}^{t} \Phi_\Sigma(t, \sigma(\tau))u(\tau)\triangle\tau, \tag{10}$$

where $\Phi_\Sigma(t, s) = Ce_A(t, s)B$. The map $\mathcal{S}_\Sigma$ defined by

$$\mathcal{S}_\Sigma(u) = y,$$

where u and y are input (control) and output (observation), respectively, related by (10), is called the *input/output map* of the system Σ.

Now let us assume that $\mathcal{S}$ is an abstract input/output map acting on the input u as follows

$$\mathcal{S}(u)(t) = \int_{t_0}^{t} \Phi(t, \sigma(\tau))u(\tau)\triangle\tau, \tag{11}$$

where $\Phi : (t, s) \mapsto \Phi(t, s) \in \mathbb{R}^{r\times m}$, is defined for all $t \geq s \geq t_0$. To have the integral well defined we assume that for any $t > t_0$, the map $s \mapsto \Phi(t, s)$ is rd-continuous.

We are interested in finding a state-space representation of $\mathcal{S}$, i.e. we are looking for a system Σ, described by constant matrices A, B and C, such that the maps $\mathcal{S}$ and $\mathcal{S}_\Sigma$ coincide. This is equivalent to finding representation for the kernel Φ of the integral operator $\mathcal{S}$, of the form

$$\Phi(t, s) = Ce_A(t, s)B$$

for $t_0 \leq s \leq t$. It should be obvious that Φ must satisfy some conditions to have such representation. We want to find these conditions and construct matrices A, B and C, when they exist.

The problem stated above has known solutions for two standard time scales: $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

If $\mathbb{T} = \mathbb{R}$ then $\Phi_\Sigma(t, s) = Ce^{A(t-s)}B$. Thus it is enough to study the function $\Psi_\Sigma(t) = Ce^{At}B$, as

$$\Phi_\Sigma(t, s) = \Psi_\Sigma(t - s).$$

The (standard) derivatives of Ψ_Σ at $t_0 = 0$ are called *Markov parameters* of the system Σ. It is clear that $\Psi_\Sigma^{(k)}(0) = CA^kB$.

If now the abstract input/output map $\mathcal{S}$ is a convolution operator

$$\mathcal{S}(u)(t) = \int_0^t \Psi(t - \tau)u(\tau)d\tau$$

whose kernel matrix Ψ is assumed to be analytic, then the *Markov parameters* of the map $\mathcal{S}$ are defined as

$$M_k = \Psi^{(k)}(t_0),\ k \geq 0.$$

The realization problem of the input-output map reduces now to finding, if possible, matrices A, B and C such that $M_k = CA^kB$. It is well known that such matrices exist if and only if the Markov parameters M_k, $k \geq 0$, satisfy a linear recurrence condition of the form

$$M_{p+k} + a_{p-1}M_{p-1+k} + \ldots + a_1M_{1+k} + a_0M_k = 0 \tag{12}$$

for some $a_0, a_1, \ldots, a_{p-1} \in \mathbb{R}$ and all $k \geq 0$.

If $\mathbb{T} = \mathbb{Z}$ then the system Σ takes on the form

$$x(t + 1) - x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t),\ t \in \mathbb{Z}$$

and similar calculations lead to the following form of Ψ_Σ

$$\Psi_\Sigma(t) = C(I + A)^tB, \quad t \in \mathbb{Z}_+.$$

Now the delta derivative is the forward difference, so

$$\Psi_{\Sigma}^{\triangle^k}(t) = C(I+A)^t A^k B,$$

and once again the Markov parameters of Σ become CA^kB, $k \geq 0$. The solution of the problem is then exactly the same as in the continuous-time case.

Remark 5.1. Usually a discrete-time system is described by a difference equation of the form:

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad t \in \mathbb{Z}_+. \tag{13}$$

If we assume a zero initial condition, then the input/output map is given by the discrete convolution

$$y(t+1) = \sum_{k=0}^{t} CA^{t-k}Bu(k),$$

of the sequences $(CA^kB)_{k\geq 0}$ and $(u(k))_{k\geq 0}$. Thus the Markov parameters CA^kB show up directly in the description of the input/output map of the system.

The cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ may be easily unified and extended to the case of *homogeneous* time scales, for which μ is constant. For general time scales the theory is more complicated as $\Phi_\Sigma(t,s)$ cannot be expressed as a function of $t-s$. Nevertheless, basic ideas may be transferred to an arbitrary time scale.

Let Σ be a system of the form (4). Then $\Phi_\Sigma(t,s) = Ce_A(t,s)B$. Let $\Phi_\Sigma^{t_0}(t) := \Phi_\Sigma(t,t_0)$. Then $\Phi_\Sigma^{t_0}(t) = Ce_A^{t_0}(t)B$, so $\Phi_\Sigma^{t_0}$ is infinitely many times differentiable at t_0. We now define the Markov parameters:

$$M_k := (\Phi^{t_0})^{\triangle^k}(t_0).$$

It is clear that $M_k = CA^kB$. Observe that the Markov parameters do not depend on t_0.

Let $\mathcal{S}$ be an input/output map given by (11). To find a realization of $\mathcal{S}$ one needs to find Σ such that $\Phi = \Phi_\Sigma$. The following theorem gives necessary and sufficient conditions for existence of realization. One can see it as an extension of a similar result in [4]. The proof of the 'if' part contains a construction of matrices A, B and C that define Σ.

Theorem 5.2. *There exists a system Σ such that $\Phi = \Phi_\Sigma$ if and only if for each $t_1 \in \mathbb{T}$, $t_1 \geq t_0$, there exist matrix valued delta differentiable functions $G_{t_1} : [t_1, \infty) \to \mathbb{R}^{r\times n}$ and $H_{t_1} : [t_0, t_1] \to \mathbb{R}^{n\times m}$, such that the following conditions hold:*

i) *for all $t_1 > \sigma^{n-1}(t_0)$ the matrix $\int_{t_0}^{t_1} H_{t_1}(\sigma(s))H_{t_1}^T(\sigma(s))\triangle s$ is invertible and the matrices $(\int_{t_0}^{t_1} H_{t_1}^{\triangle}(s)H_{t_1}^T(\sigma(s))\triangle s)(\int_{t_0}^{t_1} H_{t_1}(\sigma(s))H_{t_1}^T(\sigma(s))\triangle s)^{-1}$, $H_{t_1}(t_1)$ and $G_{t_1}(t_1)$ do not depend on t_1;*

ii) *$\Phi(t,s) = G_{t_1}(t)H_{t_1}(s)$ for any $t_1 \geq t_0$ and $s \leq t_1 \leq t$, $t, t_1, s \in \mathbb{T}$;*

iii) *for any $t_1 \in \mathbb{T}$ and $s \leq t_1 \leq t$: $G_{t_1}^{\triangle}(t)H_{t_1}(\sigma(s)) + G_{t_1}(t)H_{t_1}^{\triangle}(s) = 0$.*

Proof. Necessity. Let $G_{t_1}(t) := Ce_A(t,t_1)$ for $t \geq t_1$ and $H_{t_1}(s) := e_A(t_1,s)B$ for $s \leq t_1$. Then for $s \leq t_1 \leq t$

$$\Phi(t,s) = Ce_A(t,t_1)e_A(t_1,s)B = G_{t_1}(t)H_{t_1}(s) \tag{14}$$

so ii) is satisfied. iii) follows from the fact that $(e_A^{t_1})^{\triangle}(t) = Ae_A^{t_1}(t)$ and $({}_{t_1}e_A)^{\triangle}(s) = -{}_{t_1}e_A(\sigma(s))A$ and from the commutativity condition $Ae_A(t,s) = e_A(t,s)A$. Invertibility of the matrix $\int_{t_0}^{t_1} H_{t_1}(\sigma(s))H_{t_1}^T(\sigma(s))\triangle s$ in the condition i) is equivalent to controllability (see [2]) of Σ and, in general, may not be satisfied. Then the standard procedure is to restrict the system to the reachable set from 0, which is a linear subspace of the state space. The input/output map does not change and the reduced system becomes controllable, so the modified matrix $\int_{t_0}^{t_1} H_{t_1}(\sigma(s))H_{t_1}^T(\sigma(s))\triangle s$ becomes invertible. A simple computation shows that the other matrices that appear in i) are equal, respectively, to $-A$, B and C, so they do not depend on t_1.

Sufficiency. Let us assume that $\Phi(t,s) = G_{t_1}(t)H_{t_1}(s)$ for any $t_1 \geq t_0$ and $s \leq t_1 \leq t$. Then, multiplying the equation in iii) by $H_{t_1}^T(\sigma(s))$ and integrating with respect to s over the interval $[t_0,t_1]$, we get

$$G_{t_1}^{\triangle}(t)\int_{t_0}^{t_1} H_{t_1}(\sigma(s))H_{t_1}^T(\sigma(s))\triangle s = -G_{t_1}(t)\int_{t_0}^{t_1} H_{t_1}^{\triangle}(s)H_{t_1}^T(\sigma(s))\triangle s$$

Define (for $t_1 > \sigma^{n-1}(t_0)$)

$$A := -(\int_{t_0}^{t_1} H_{t_1}^{\triangle}(s)H_{t_1}^T(\sigma(s))\triangle s)(\int_{t_0}^{t_1} H_{t_1}(\sigma(s))H_{t_1}^T(\sigma(s))\triangle s)^{-1}$$

Then G_{t_1} satisfies $G_{t_1}^{\triangle}(t) = G_{t_1}(t)A$ so $G_{t_1}(t) = G_{t_1}(t_1)e_A(t,t_1)$. Then, substituting G_{t_1} to iii) we have

$$G_{t_1}(t_1)e_A(t,t_1)AH_{t_1}(\sigma(s)) + G_{t_1}(t_1)e_A(t,t_1)H_{t_1}^{\triangle}(s) = 0$$

and

$$G_{t_1}(t_1)e_A(t,t_1)[AH_{t_1}(\sigma(s)) + H_{t_1}^{\triangle}(s)] = 0 \tag{15}$$

The solution of $AH_{t_1}(\sigma(s)) + H_{t_1}^{\triangle}(s) = 0$ is $H_{t_1}(s) = e_A(t_1,s)H_{t_1}(t_1)$.

Defining $B := H_{t_1}(t_1)$ and $C := G_{t_1}(t_1)$ we have $\Phi(t,s) = Ce_A(t,s)B$ for all $s \leq t_1 \leq t$. As this holds for arbitrary $t_1 \geq t_0$ and A, B and C do not depend on t_1, this equation is satisfied for all $t \geq s \geq t_0$. □

Assume now that Φ^s is infinitely many times differentiable at s for every $s \geq t_0$, Let us define the Markov parameters of the input/output map $\mathcal{S}$ as:

$$M_k^s := (\Phi^s)^{\triangle^k}(s)$$

where $s \geq t_0$.

To express necessary and sufficient conditions for existence of realization using the Markov parameters, we need a stronger assumption on Φ. We say that Φ *is analytic at* $s \geq t_0$, if Φ^s is uniquely determined by M_k^s, $k \geq 0$, and that Φ is analytic, if Φ^s is analytic at every s.

Remark 5.3. Usually analyticity means that the function can be expanded in a power series. This implies the property we need, i.e. derivatives at some point determine (at least locally) the function. Though power functions and polynomials on time scales are well defined (see [3]), to our knowledge, the theory of power series on time scales does not exists. Once such a theory is developed, the concept of analyticity defined above could be made more standard. For $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$ power series and analyticity are either well known or easy to define. This can be extended to time scales that are unions of closed intervals and discrete sets.

Theorem 5.4. *Assume that Φ defining $\mathcal{S}$ is analytic. The input/output map $\mathcal{S}$ has a realization if and only if the Markov parameters M_k^s do not depend on s and there are $p \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{p-1} \in \mathbb{R}$ such that for all $k \geq 0$*

$$M_{p+k} + a_{p-1}M_{p-1+k} + \ldots + a_1 M_{1+k} + a_0 M_k = 0. \tag{16}$$

Proof. If the input/output map $\mathcal{S}$ has a realization, then $\Phi(t,s) = Ce_A(t,s)B$ for some matrices A, B and C. Thus $M_k^s = CA^kB$, they do not depend on s and (16) follows from the Cayley-Hamilton theorem for A.

On the other hand, if (16) holds, then for $A = \begin{pmatrix} 0_r & 0_r & \ldots & -a_0 I_r \\ I_r & 0_r & \ldots & -a_1 I_r \\ 0_r & I_r & \ldots & -a_2 I_r \\ \vdots & \vdots & \vdots & \vdots \\ 0_r & 0_r & \ldots & -a_{p-1} I_r \end{pmatrix}$,

$B = \begin{pmatrix} I_r \\ 0_r \\ \vdots \\ 0_r \end{pmatrix}$ and $C = (M_0, M_1, \ldots, M_{p-1})$ we obtain $M_k = CA^kB,\ k \geq 0$.

Analyticity of Φ implies that $\Phi(t,s) = Ce_A(t,s)B$ for all $t \geq s \geq t_0$. This means that A, B and C define a realization of $\mathcal{S}$. □

Example 5.5. Let $\mathbb{T} = \bigcup_{k \in \mathbb{Z}} [2k, 2k+1]$. For $s \in [2l, 2l+1]$ and $t \in [2k, 2k+1]$ let

$$\Phi(t,s) = \Phi^s(t) = \begin{cases} e^{-(t-s)} - e^{-2(t-s)} & \text{for } k = l \\ e^{-2(t-s-k+l)}(-1)^{k-l+1} & \text{for } k \neq l \end{cases}$$

Then, for any $s \leq t < 2l+1$ and $i \in \mathbb{N}$ we have

$$(\Phi^s)^{\Delta^i}(t) = (-1)^i e^{-(t-s)} - (-2)^i e^{-2(t-s)}$$

so for $s < 2l+1$, $M_i^s = (-1)^i - (-2)^i = (-1)^i(1 - 2^i)$.

Now let $s = 2l+1$. Then for $2l+2 \leq t < 2l+3$

$$(\Phi^{2l+1})^{\Delta^i}(t) = (-2)^i e^{-2(t-2l-2)},$$

so, in particular, $(\Phi^{2l+1})^{\Delta^i}(2l+2) = (-2)^i$. Hence, by induction, we get $(\Phi^{2l+1})^{\Delta^i}(2l+1) = (-1)^i(1-2^i)$, which gives that for every $s \in \mathbb{T}$

$$M_i^s = (-1)^i(1-2^i).$$

Thus M_i^s does not depend on s. Moreover the Markov parameters satisfy the linear recurrence condition $M_{i+2} + 3M_{i+1} + 2M_i = 0$. Hence the realization is given by the matrices

$$A = \begin{pmatrix} 0 & -2 \\ 1 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = (0,1)$$

Observe that the realization is controllable and observable. One can also find the matrices $G_{t_1}(t)$ and $H_{t_1}(s)$, which factorize $\Phi(t,s)$ for each $t_1 \in [s,t]$. They are given as follows:

$$G_{t_1}(t) = (e^{-(t-t_1)} - e^{-2(t-t_1)}, -e^{-(t-t_1)} + 2e^{-2(t-t_1)}),$$
$$H_{t_1}(s) = \begin{pmatrix} 2e^{-(t_1-s)} - e^{-2(t_1-s)} \\ e^{-(t_1-s)} - e^{-2(t_1-s)} \end{pmatrix}$$

for $k = l$ and

$$G_{t_1}(t) = (-1)^{k-p-1}(e^{-2(t-t_1-k+p)}, -2e^{-2(t-t_1-k+p)}),$$
$$H_{t_1}(s) = (-1)^{p-l-1}\begin{pmatrix} e^{-2(t_1-s-p+l)} \\ e^{-2(t_1-s-p+l)} \end{pmatrix}$$

for $t_1 \in [2p, 2p+1]$ and $l < p < k$. Similarly for the cases $l = p < k$ and $l < p = k$.

Example 5.6. Let $\mathbb{T} = \bigcup_{k\in\mathbb{Z}} [2k, 2k+1]$ and let

$$\Phi^s(t) = e^{-(t-s-k+l)}(-2)^{k-l}$$

for $t \in [2k, 2k+1]$ and $s \in [2l, 2l+1]$. Then, for any $s \le t < 2l+1$ we have $(\Phi^s)^\Delta(t) = -e^{-(t-s)}$, so $(\Phi^s)^\Delta(s) = -1$, and for $s = 2l+1$ we have $(\Phi^s)^\Delta(s) = -3$. This means that the Markov parameters depend on s, so a realization does not exist.

As usual, realizations are not unique. From a practical point of view one would like to have a realization for which the state space has the minimal possible dimension. This is equivalent to finding a minimal realization (i.e. controllable and observable).

Theorem 5.7. *For matrices $A \in M_{n\times n}$, $B \in M_{n\times m}$, $C \in M_{r\times n}$ there exist a natural number $\tilde{n}$ and matrices $\tilde{A} \in M_{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in M_{\tilde{n}\times m}$, $\tilde{C} \in M_{r\times\tilde{n}}$ such that $Ce_A(t,t_0)B = \tilde{C}e_{\tilde{A}}(t,t_0)\tilde{B}$ for any $t \in [t_0,+\infty)$ and the pair $(\tilde{A},\tilde{B})$ is controllable and the pair $(\tilde{A},\tilde{C})$ is observable.*

Proof. The classical proof works well for all times scales. Let us assume that the pair (A, B) is not controllable and $\operatorname{rank}[A|B] = l < n$. It is known that there exists a nonsingular matrix $P \in M_{n\times n}$ such that $PAP^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ and $PB = \begin{bmatrix} B_{11} \\ 0 \end{bmatrix}$, $A_{11} \in M_{l\times l}$, $A_{22} \in M_{(n-l)\times(n-l)}$, $B_1 \in M_{l\times m}$, and the pair (A_{11}, B_1) is controllable. For $t \in [t_0, +\infty)$ let

$$Pe_A(t, t_0)P^{-1} = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ 0 & S_{22}(t) \end{bmatrix} := \begin{bmatrix} e_{A_{11}}(t, t_0) & e_{A_{12}}(t, t_0) \\ 0 & e_{A_{22}}(t, t_0) \end{bmatrix}$$

Then

$$Ce_A(t, t_0)B = CP^{-1}Pe_A(t, t_0)P^{-1}PB$$
$$= [C_1, C_2] \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ 0 & S_{22}(t) \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = C_1S_{11}(t)B_1$$

for some matrix C_1.

If the pair (A, C) is not observable, then in a similar way one can show that $Ce_A(t, t_0)B = C_1e_{A_{11}}(t, t_0)B_1$ for some matrices A_{11}, B_1, C_1, any $t \in [t_0, +\infty)$, and the pair (A_{11}, C_1) being observable. □

Corollary 5.8. *For any function Ψ there exists its minimal realization (A, B, C) such that the pairs (A, B) and (A, C) are respectively controllable and observable.*

Proposition 5.9. *If Σ and $\tilde{\Sigma}$ are two minimal realizations of the same input/output map, then the dimensions of their state spaces are equal.*

This is a standard fact for continuous time systems. The proof is algebraic and the time scale does not appear in it. One can show that the dimension of a minimal realization is equal to the rank of the Hankel matrix associated to the input-output map

$$\begin{pmatrix} M_0 & M_1 & M_2 & \cdots \\ M_1 & M_2 & M_3 & \cdots \\ M_2 & M_3 & M_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Definition 5.10. Let Σ and $\tilde{\Sigma}$ be two systems given by matrices A, B, C and $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, respectively. Assume that for both systems the state space is $\mathbb{R}^n$, the space of input values is $\mathbb{R}^m$ and the space of output values is $\mathbb{R}^r$. We say that Σ *and* $\tilde{\Sigma}$ *are isomorphic* if there is a nonsingular $n \times n$ matrix S such that $\tilde{A} = S^{-1}AS$, $\tilde{B} = S^{-1}B$ and $\tilde{C} = CS$.

Remark 5.11. An isomorphism of systems corresponds to a linear change of coordinates of the state space defined by the matrix S. Isomorphism of systems is an equivalence relation in the set of all systems with the same dimensions of states, inputs and outputs.

Proposition 5.12. *Two isomorphic systems have the same input/output map.*

Proof. Let $\tilde{A} = S^{-1}AS$, $\tilde{B} = S^{-1}B$ and $\tilde{C} = CS$ for a nonsingular matrix S. Then $S^{-1}(e_A^{t_0})^{\triangle}(t)S = S^{-1}ASS^{-1}e_A(t,t_0)S$. This means that $e_{\tilde{A}}(t,t_0) = S^{-1}e_A(t,t_0)S$ and $\tilde{C}e_{\tilde{A}}(t,t_0)\tilde{B} = CSS^{-1}e_A(t,t_0)SS^{-1}B = Ce_A(t,t_0)B$. This gives equality of the input/output maps of the two systems. □

Theorem 5.13. *If Σ and $\tilde{\Sigma}$ are minimal realizations of the same input/output map, then they are isomorphic.*

The proof of Theorem 5.13 is exactly the same as in the classical case $\mathbb{T} = \mathbb{R}$, because only linear algebra is used.

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