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GDQ Criteria of Viability for Differential Inclusions

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Abstract— **The viability problem for differential inclusions is studied. It is assumed that the right-hand side of the differential inclusion is given by a multifunction (an orientor field) defined by the graph of another multifunction (called a constraint multifunction), which depends on time. We use Generalized Differential Quotients as a differentiation tool in tangential condition. We assume that the constraint multifunction has a GDQ-regular multiselection and that the orientor field is upper semi-continuous with respect to the state variable. We also impose some weak measurability conditions. In order to formulate the main viability theorem we present some auxiliary results on Cellina continuously approximable multifunctions and Generalized Differential Quotients. The main result states that the differential inclusion has a global solution.**

Index Terms— **viability, generalized differential quotients, CCA set-valued maps**

I. Motivation and introduction

A. Motivation

Viability theory has its origin in the statement of Nagumo who solved in 1942 the viability problem for differential equations. He formulated necessary and sufficient conditions guaranteing that the solution to a Cauchy problem will stay in a specified closed set.

Theorem 1: (Nagumo, 1942) Let K_0 be a closed subset of $\mathbb{R}^n$ and let $f : K_0 \to \mathbb{R}^n$ be a continuous, bounded map. A necessary and sufficient condition for a differential equation $\dot{y} = f(y)$ to have a viable solution for any initial condition $y_0 \in K_0$ is

$$f(y) \in \left\{ w \in \mathbb{R}^n : \liminf_{t \downarrow 0} \frac{dist(y + tw, K_0)}{t} = 0 \right\}$$

for any $y \in K_0$.

For a set-valued case to solve the viability problem means to find necessary or/and sufficient condition that guarantee the existence of at least one solution staying in the graph of the constraint multifunction. In other words, viability theorems yield selection procedure of viable evolutions. This is very useful in many branches of science as, for example, in economics (prices or other fiduciary goods), genetics (genotypes or fitness matrices), medicine (affinity matrices in immunological systems), differential games (strategies) or even in sociology (cultural codes, when the evolution of societies is regulated by every individual believing and obeying such codes) (for more examples see [1]). It turns out that not all evolutions, for variety reasons, are possible in these systems. Being more precise, the state of the system must obey constraints, called viability constraints, such as homeostatic constraints in biological regulation, scarcity constraints in economics, state constraints in control, power constraints in game theory, ecological constraints in genetics, sociability constraints in sociology, etc. Thus it is necessary to select solutions which are viable in the sense that they satisfy these constraints. Viability was studied by many mathematicians. Let us mention a few of them: Jean Pierre Aubin, Helene Frankowska, Tadeusz Rzeżuchowski, Sławomir Plaskacz, Francis Clarke, Yu. S. Ledyaev, Nikolas Papageorgiou, Shouchuan Hu, Peter Tallos.

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B. Introduction

A multifunction (a set-valued map) F from a set X to a set Y assigns to every $x \in X$ a subset, maybe empty, of Y. We denote this by $F : X \twoheadrightarrow Y$ or $X \ni x \mapsto\!\!\!\!\rightarrow F(x) \subseteq Y$. The domain of F is $Do(F) = \{x \in X : F(x) \neq \emptyset\}$ and the graph of F is the set $Gr(F) = \{(x, y) \in X \times Y : y \in F(x)\}$. By $SVM(X, Y)$ we shall denote the set of all set-valued maps from X to Y. In applications one often wants to control the infinitesimal behavior of the multifunction around a point (x, y) belonging to its graph. For this some generalization of ordinary derivative is needed. We recall here two concepts: more classical contingent derivative (see [1], [2], [3], [4]) and newer Generalized Differential Quotient (GDQ) introduced by H. Sussmann (see [20], [21]). Both can be also used for single-valued functions that do not have ordinary derivatives. We show relations between these concepts for multifunctions from $\mathbb{R}$ to $\mathbb{R}^n$. Such multifunctions appear as constraints in viability problems.

Let $K : [0, 1] \twoheadrightarrow \mathbb{R}^n$ and $F : Gr(K) \twoheadrightarrow \mathbb{R}^n$. Then F is called a time dependent orientor field with the restriction multifunction K. Consider the initial value problem for F:

$$\dot{x}(t) \in F(t, x(t)), \quad x(t_0) = x_0, \tag{1}$$

where $(t_0, x_0) \in Gr(K)$. We say that F is *viable* if for every $(t_0, x_0) \in Gr(K)$ (1) has an absolutely continuous solution $x : [t_0, 1] \to \mathbb{R}^n$ such that $x(t) \in K(t)$ for all $t \in [t_0, 1]$ and the inclusion in (1) is satisfied almost everywhere on $[t_0, 1]$.

Many different criteria of viability can be found in the literature. The reader can consult e.g. [1], [3], [5], [7], [8], [9], [10], [13], [17], [18], [22] for the results and properties used in the statements (like upper semi-continuity or ε-δ-usc from the left). The authors impose various conditions on K and F and propose some tangency requirements, which

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generalize the original condition given by Nagumo for time independent vector fields.

In the tangency condition usually the contingent derivative of K is used. In [13] and in this paper, we prefer to use GDQ of K. We feel that it is more natural than the contingent derivative and GDQ-differentiability implies other important properties of K that are used in the viability theorem — the main result of this paper. It extends the result of [13] by relaxing some restrictions imposed on F.

In Section II we start with some basic definitions that are needed for further results. And so, we present the definition of Cellina continuously approximable set-valued maps (abbreviated CCA). This concept, which is a substitution of continuity for multifunctions, was studied first by Arrigo Cellina in 70's and later by Hector Sussmann in 90's (firstly he called CCA set-valued maps 'regular set-valued maps'). To make this matter more clear for a reader we put some theorems, examples and illustrations. In this section there is also presented the definition of Generalized Differential Quotients (abbr. GDQs) as well as some important theorems about them. Relations with another generalized differentiation theories and examples complete the picture of GDQs. Some other definitions that are needed for the main result are also included in the section. Finally, the Jarnik-Kurzweil Theorem that is crucial in the proof of the viable theorem, ends the section.

Section III, that is the most important in this paper, is devoted to viability theory. At the beginning of the section we set the problem. Then a proposition on viability for the right hand side of differential inclusion that is 'almost u.s.c.' is given. The main content of this section is a viability theorem. Some important remarks and example finish the paper.

II. Basic definitions

A. Cellina continuously approximable set-valued maps

Let X and Y be metric spaces. We say that a set-valued map $F : X \twoheadrightarrow Y$ is *upper semicontinuous* (abbr. u.s.c.) at $\bar{x} \in Do(F)$ if and only if for any neighborhood U of $F(\bar{x})$ there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$, $F(x) \subset U$.

Let $A, B \subseteq X$. Then

$$\triangle(A, B) = \sup\{dist(q, B) : q \in A\}$$

is called the *semi-distance* between sets A and B where $dist(a, B) := \inf_{b \in B} d(a, b)$

Definition 1: We say that a sequence of set-valued maps $F_n : X \twoheadrightarrow Y$, $n \in \mathbb{N}$, *graph-converges* to a set-valued map $F : X \twoheadrightarrow Y$, and write $F_n \xrightarrow{gr} F$, if

$$lim_{n \to \infty} \triangle(Gr(F_n), Gr(F)) = 0.$$

Definition 2: A set-valued map $F : X \twoheadrightarrow Y$ is called *Cellina continuously approximable* (abbr. CCA) if for every compact subset K of X

(1) $Gr(F\,|_K)$ is compact;
(2) there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of single-valued continuous maps $f_j : K \to Y$ that graph-converges to $F\,|_K$.

By $CCA(X, Y)$ we shall denote the set of all set-valued maps from X to Y. The CCA property is a kind of substitute of continuity for set-valued map. This is illustrated by the following theorems.

Theorem 2: [21] Assume that K is a compact metric space, Y is a normed space, and C is a convex subset of Y. Let $F \in SVM(K, C)$ be a set-valued map such that the graph of F is compact and the value $F(x)$ is a nonempty convex set for every $x \in K$. Then F is CCA as a map from K to C.

Theorem 3: [21] Assume that X is a metric space, Y is a normed space, and C is a convex subset of Y. Let $F \in SVM(X, C)$ be an upper semicontinuous set-valued map with nonempty compact convex values. Then $F \in CCA(X, C)$.

Theorem 4: [21] Assume that X, Y, Z are metric spaces. Let $F \in CCA(X, Y)$, $G \in CCA(Y, Z)$. Then the composite map $G \circ F$ belongs to $CCA(X; Z)$.

Example 1: Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ such that

$$F(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$$

To show that F is CCA it is enough to approximate F, in the graph sense, by a piecewise linear single-valued function

$$f(x) = \begin{cases} -1 & \text{if } x < -\epsilon \\ \frac{x}{\epsilon} & \text{if } -\epsilon \le x \le \epsilon \\ 1 & \text{if } x > \epsilon \end{cases}$$

where ϵ is sufficiently small.

B. Generalized Differential Quotients

Definition 3: Let $m, n \in \mathbb{Z}_+$, $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ be a set-valued map, $\overline{x} \in \mathbb{R}^m$, $\overline{y} \in \mathbb{R}^n$, $\overline{y} \in F(\overline{x})$ and let Λ be a nonempty compact subset of $\mathbb{R}^{n \times m}$. Let S be a subset of $\mathbb{R}^m$. Then an element of Λ is an $n \times m$ matrix. We say that Λ is a *generalized differential quotient (GDQ) of F at $(\overline{x}, \overline{y})$ in the direction S*, and write $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ if for every positive real number δ there exist U, G such that
1. U is a compact neighborhood of 0 in $\mathbb{R}^m$ and $U \cap S$ is compact;
2. G is a CCA set-valued map from $\overline{x} + U \cap S$ to the δ-neighborhood Λ^δ of Λ in $\mathbb{R}^{n \times m}$;
3. $G(x) \cdot (x - \overline{x}) \subseteq F(x) - \overline{y}$ for every $x - \overline{x} \in U \cap S$

A multifunction F may have many GDQs at $(\overline{x}, \overline{y})$. In particular, if $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ then for any $\tilde{\Lambda} \supset \Lambda$ also $\tilde{\Lambda} \in GDQ(F; \overline{x}, \overline{y}; S)$. F is called *GDQ-differentiable* if the set $GDQ(F; \overline{x}, \overline{y}; S)$ is not empty. We have the following:

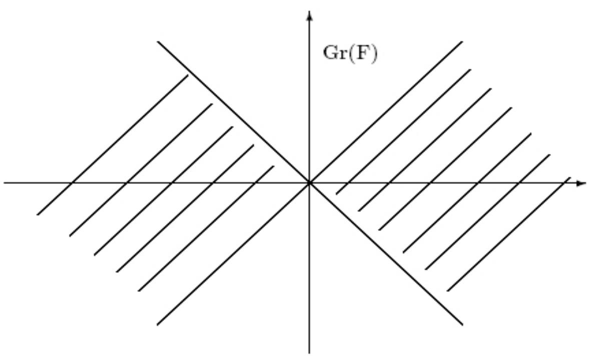
Theorem 5 (Minimality Theorem): [12] If the set of GDQs of a set-valued map F at a point $(\overline{x}, \overline{y})$ in the direction S is not empty, then there exists in this set at least one minimal GDQ at this point in the direction S in the sense of inclusion of sets.

Corollary 1: Every element Λ of $GDQ(F; \overline{x}, \overline{y}; S)$ contains a minimal element of $GDQ(F; \overline{x}, \overline{y}; S)$ in the sense of inclusion of sets.

Example 2: Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map such that

$$F(x) = \begin{cases} [-|x|, |x|] & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \end{cases}$$

Fig. 1. The graph of F.



Then any singleton $\{a\}$ for $a \in [-1, 1]$ is a minimal GDQ of F at the point $(0, 0)$.

The following proposition is very useful in the proof of the main result of this paper.

Proposition 1: If K is GDQ-differentiable at (s, y) in the direction $\mathbb{R}_+$, then there exists a measurable map $\gamma : [s, s+\delta] \to \mathbb{R}^n$ such that $\gamma(t) \in K(t)$ for $t \in [s, s+\delta]$, $\gamma(s) = y$ and γ is continuous at s.

C. GDQs and the contingent derivative

Let X be a normed space. Recall that the *contingent cone* (the "Bouligand cone") to a set $C \subset X$ at x is defined by

$$T_C(x) = \left\{ w \in X : \liminf_{t \downarrow 0} \frac{dist(x + tw, C)}{t} = 0 \right\}.$$

Definition 4: Let $F : K \twoheadrightarrow Y$, $K \subset X$, and $F(x)$ be nonempty for all $x \in K$. The *contingent derivative* $DF(x_0, y_0)$ of F at $x_0 \in K$ and $y_0 \in F(x_0)$ is a set-valued map from X to Y whose graph is the *contingent cone* $T_{Gr(F)}(x_0, y_0)$ to the graph of F at (x_0, y_0).

In other words, $v_0 \in DF(x_0, y_0)(u_0) \iff (u_0, v_0) \in T_{Gr(F)}(x_0, y_0)$.

Theorem 6: [13] Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$, $Do(F) = \mathbb{R}$, and $\Lambda \in GDQ(F; x_0, y_0; \mathbb{R}_+)$. If Λ is minimal, then $\Lambda \subseteq DF(x_0, y_0)(1)$.

Corollary 2: Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$. If F is GDQ-differentiable at (x, y) in the direction $\mathbb{R}_+$ ($\mathbb{R}_-$) then there exists the contingent derivative $DF(x, y)(1)$ ($DF(x, y)(-1)$).

D. Auxiliary definitions and results

Definition 5: Let $T = [a, b]$. We say that $F : G \twoheadrightarrow \mathbb{R}^n$, where $G \subset T \times \mathbb{R}^n$, is a *Scorza-Dragoni set-valued map* if for every $\varepsilon > 0$ there exists a closed set $T_\varepsilon \subset T$ such that $\lambda(T \backslash T_\varepsilon) \leq \varepsilon$ and the multifunction $(t, y) \mapsto F(t, y)$ is upper semi-continuous (u.s.c.) on $(T_\varepsilon \times \mathbb{R}^n) \cap G$.

Let us define a set-valued map $SGDQ(K; t, x; \mathbb{R}_+)$ as the closure of the union of all minimal GDQs of K at $(t, x) \in Gr(K)$ in the direction $\mathbb{R}_+$.

We say that $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T$, is *ε-δ-u.s.c. from the left* if for every $t_0 \in (0, a]$ and $\varepsilon > 0$ there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$K(t) \subseteq K(t_0) + \varepsilon B(0)$$

for all $t \in (t_0 - \delta, t_0] \cap T$.

Definition 6: Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $T = [a, b]$ for $a, b \in \mathbb{R}$. We say that K is *GDQ-regular* if

1) K is GDQ-differentiable in the direction $\mathbb{R}_+$ at every $t \in [a, b)$ and every $y \in K(t)$
2) K is ε-δ-u.s.c form the left on T
3) K has closed values
4) $(t, x) \mapsto SGDQ(K; t, x; \mathbb{R}_+)$ is a Scorza-Dragoni set-valued map.

The following theorem is often used in the proofs of viability theorems. It allows to reduce the original problem with weak assumptions on F to another one, where a modified orientor field F_0 is Scorza-Dragoni.

Theorem 7 (Jarnik-Kurzweil, [16]): Assume that $G \subseteq \mathbb{R} \times Y$, $F : G \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = G$, and that for almost all t the map $F(t, \cdot)$ is upper semicontinuous with compact convex values. Then there exists a Scorza-Dragoni multifunction $F_0 : G \twoheadrightarrow \mathbb{R}^n$ with compact convex values satisfying $F_0(t, y) \subseteq F(t, y)$ for $(t, y) \in G$, and such that if $T \subset \mathbb{R}$ is measurable , $u : T \to Y$ and $v : T \to \mathbb{R}^n$ are measurable maps such that $v(t) \in F(t, u(t))$, for almost all $t \in T$, then $v(t) \in F_0(t, u(t))$ for almost all $t \in T$.

III. Viability results

Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = [0, 1] = T \subset \mathbb{R}$, be a constraint multifunction and $F : Gr(K) \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = Gr(K)$, be an orientor field (i.e. multivalued vector field). Consider the multivalued Cauchy problem as follows:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)), & \text{a.e. on } T \\ x(t_0) = x_0. \end{cases} \tag{2}$$

The following proposition shows the existence of a solution to (2), under GDQ-differentiability condition on K.

Proposition 2: [13] Let $K : T \twoheadrightarrow \mathbb{R}^n$ be GDQ-differentiable at every $(t, y) \in GrK$ in the direction $\mathbb{R}_+$. Assume that K is ε-δ-u.s.c. from the left with closed values. Let $F : GrK \twoheadrightarrow \mathbb{R}^n$, with $Do(F) = GrK$, be a multifunction with closed convex values such that

1) for every $\varepsilon > 0$ there is a closed set $T_\varepsilon \subseteq T$ with $\lambda(T \backslash T_\varepsilon) \leq \varepsilon$ such that $F |_{(T_\varepsilon \times \mathbb{R}^n) \cap Gr(K)}$ is u.s.c.;
2) there exists $N \subset [0, 1]$, $\lambda(N) = 0$, such that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ for all $(t, y) \in ((T \backslash N) \times \mathbb{R}^n) \cap Gr(K)$, and
3) $||F(t, y)|| \leq a(t)(1 + ||y||)$ for almost every $t \in T$ and for every $y \in K(t)$, with $a \in L^1(T)$,

where λ is the Lebesgue measure. Then (2) has a solution $y : [t_0, 1] \to \mathbb{R}^n$, which is an absolutely continuous function.

We impose the following assumptions on K and F:

H(K): for any $t_0 \in [0, 1)$ and $x_0 \in K(t_0)$ there exists $K_{(t_0,x_0)} : [t_0, 1] \twoheadrightarrow \mathbb{R}^n$ such that $K_{(t_0,x_0)}$ is GDQ-regular and $K_{(t_0,x_0)}(t) \subseteq K(t)$ for every $t \in [t_0, 1]$.

H(F):
(i) $F : GrK \twoheadrightarrow \mathbb{R}^n$ has closed convex values;
(ii) for any measurable $\gamma(\cdot)$ the multifunction $t \mapsto F(t, \gamma(t))$ is measurable;
(iii) for any $t_0 \in [0, 1]$ and $x_0 \in K(t_0)$ there exists $\alpha \in L^1(t_0, 1)$ such that for any $(t, x) \in GrK_{(t_0,x_0)}$, $||F(t, x)|| \leq \alpha(t)(1 + ||x||)$;
(iv) for any $(t_0, x_0) \in GrK$ and for any $t \in [t_0, 1]$ the multifunction
$x \mapsto F |_{GrK_{(t_0,x_0)}} (t, x)$ is u.s.c.

H: for any $(t_0, x_0) \in GrK$, for almost every $t \in [t_0, 1]$ and for any $x \in K_{(t_0,x_0)}(t)$,

$$F(t, x) \cap SGDQ(K_{(t_0,x_0)}; t, x; \mathbb{R}_+) \neq \emptyset.$$

Now we can state the main result of this paper.

Theorem 8: Assume that H(K), H(F) and H hold. Then for any $(t_0, y_0) \in GrK$ the multivalued Cauchy problem (2) has a solution $y : [t_0, 1] \to \mathbb{R}^n$, which is an absolutely continuous function satisfying $y(t) \in K(t)$ for every $t \in [t_0, 1]$.

Proof: The idea of the proof is to reduce the problem to the 'almost u.s.c case' (i.e. for Scorza-Dragoni right hand side of the differential inclusion (2)) studied in [5]. First let us fix $t_0 \in [0, 1]$ and $y_0 \in K(t_0)$. From H(F)(iii) it arises that F has compact values. By H(F)(i)(ii), we can apply Proposition 7. Thus there exists $\tilde{F} : GrK_{(t_0,y_0)} \twoheadrightarrow \mathbb{R}^n$ with compact convex values such that
(i) $\tilde{F}(t, y) \subseteq F(t, y)$ for all $(t, y) \in GrK_{(t_0,y_0)}$;
(ii) for every $\varepsilon > 0$ there exists a closed set $\check{T}_\varepsilon \subseteq [t_0, 1]$ such that $\lambda([t_0, 1] \backslash \check{T}_\varepsilon) \leq \varepsilon$ and $\tilde{F} |_{(\check{T}_\varepsilon \times \mathbb{R}^n) \cap GrK_{(t_0,y_0)}}$ is u.s.c.; and
(iii) if $A \subseteq [t_0, 1]$ is measurable and $\alpha, \beta : A \to \mathbb{R}^n$ are measurable functions such that $\beta(t) \in F(t, \alpha(t))$ a.e. on A, then $\beta(t) \in \tilde{F}(t, \alpha(t))$ a.e. on A.
Thus we can apply now Proposition 2, if we prove that for almost all $t \in [t_0, 1]$ and every $y \in K(t)$, we have

$$\tilde{F}(t, y) \cap SGDQ(K_(t_0, y_0); t, y; \mathbb{R}_+) \neq \emptyset.$$

In order to prove this, we start with $\varepsilon > 0$. Let us define $T_\varepsilon := \hat{T}_{\frac{\varepsilon}{2}} \cap \check{T}_{\frac{\varepsilon}{2}}$. Then T_ε is closed and $\lambda([t_0, 1] \backslash T_\varepsilon) \leq \varepsilon$. From Lebesgue's density theorem, almost all points of T_ε are density points of T_ε. Let $\tilde{T}_\varepsilon \subseteq T_\varepsilon$ be such that $\lambda(T \backslash \tilde{T}_\varepsilon) \leq 2\varepsilon$. Hence for each $s \in \tilde{T}_\varepsilon$ there is a sequence (t_n) such that $t_n \in T_\varepsilon$, $t_n > s$ and $t_n \to s$. We can assume that $1 \notin \tilde{T}_\varepsilon$. Let $s \in \tilde{T}_\varepsilon$ and $y \in K(s)$. From GDQ-differentiability of $K_{(t_0,y_0)}$ at (s, y) in the direction $\mathbb{R}_+$ and Proposition 1, we get that there exists on $[s, s + \rho]$ $(\rho > 0)$ a single-valued map $t \to \gamma(t) \in K_{(t_0,y_0)}(t)$, measurable and continuous at s, with $\gamma(s) = y$ for $t \in [s, s+\rho]$. Because of the assumption H(F)(ii), the map $t \to F(t, \gamma(t))$ is measurable on $[s, s + \rho]$. Then $t \mapsto F(t, \gamma(t)) \cap SGDQ(K_{(t_0,y_0)}; t, \gamma(t); \mathbb{R}_+)$ is measurable on $[s, s+\rho]$, too. Moreover, from the assumption that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ for a.e. $t \in [s, s+\rho]$, it has nonempty values for a.e. $t \in [s, s + \rho]$. Now applying Proposition on measurable selection from [19], which says that a measurable set-valued map with (possibly empty) closed values has a measurable selection, there exists a measurable function $t \to y(t)$ such that $y : [s, s + \rho] \to \mathbb{R}^n$ and

$$y(t) \in F(t, \gamma(t)) \cap SGDQ(K_{(t_0,y_0)}; t, \gamma(t); \mathbb{R}_+) \tag{3}$$

for a.e. $t \in [s, s + \rho]$. By property (c) of $\tilde{F}$, we have also

$$y(t) \in \tilde{F}(t, \gamma(t)) \cap SGDQ(K_{(t_0,y_0)}; t, \gamma(t); \mathbb{R}_+)$$

for a.e. $t \in [s, s + \rho]$. Let $t_n \to s$, $t_n > s$ and $\{t_n\}_{n\geq1} \subseteq T_\varepsilon \cap S$, where $S = [s, s + \rho] \backslash N$, $\lambda(N) = 0$, and $y(t) \in F(t, \gamma(t)) \cap SGDQ(K_{(t_0,y_0)}; t, \gamma(t); \mathbb{R}_+)$ for every $t \in S$ and let $y_n := y(t_n)$. Since $\tilde{F} |_{(T_\varepsilon \times \mathbb{R}^n) \cap GrK_(t_0,y_0)}$ is u.s.c. with compact values, thus $\overline{\bigcup_{n\geq1} \tilde{F}(t_n, \gamma(t_n))}$ is compact. Therefore we may assume that $y_n \to \tilde{y}$ as $n \to \infty$ and we have $\text{Limsup}_{t_n \to s} \tilde{F}(t_n, \gamma(t_n)) \subseteq \tilde{F}(s, y)$. Then $\tilde{y} \in \tilde{F}(s, y)$. Analogously, from u.s.c. of $(s, y) \mapsto SGDQ(K_{(t_0,y_0)}; s, y; \mathbb{R}_+)$, $\tilde{y} \in SGDQ(K_{(t_0,y_0)}; s, y; \mathbb{R}_+)$. It allows to write that $\tilde{F}(t, y) \cap SGDQ(K_{(t_0,y_0)}; t, y; \mathbb{R}_+) \neq \emptyset$, for a.e. $t \in [t_0, 1]$ and every $y \in K(t)$. Then applying Proposition 2 we conclude the thesis. ■

Example 3: Consider a set-valued map $K : [-1, 0] \twoheadrightarrow \mathbb{R}$ such that

$$K(t) = \begin{cases} \{|t \sin \frac{1}{t}|\} \cup \{0\} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Let $\dot{x} \in F(t,x)$ and $x(t_0) = x_0$ where $F(t,x) = [-1 - \frac{1}{t}, 1 + \frac{1}{t}]$ and $F(0,0) = 0$ is defined on $Gr(K)$. Observe that F does not satisfy (4). But for every $t_0 \in [-1,0)$ and $x_0 \in K(t_0)$ there exists a global solution $x : [t_0, 0] \to \mathbb{R}$ defined as follows

$$x(t) = \begin{cases} t \sin \frac{1}{t} & \text{if } t \in [t_0, \tau_0] \\ 0 & \text{if } t \in (\tau_0, 0], \end{cases}$$

where $\sin \frac{1}{\tau_0} = 0$.

Observe that for K and F the assumptions of Theorem 8 are satisfied. In particular, for $t_0 \in [-1,0)$ and $x_0 \in K(t_0)$ one can take

$$K_{(t_0,x_0)}(t) = \begin{cases} \{|t \sin \frac{1}{t}|\} \cup \{0\} & \text{if } t \in [t_0, \tau_0] \\ 0 & \text{if } t \in (\tau_0, 0] \end{cases}$$

where $\sin \frac{1}{\tau_0} = 0$. Then $F \mid_{K_{(t_0,x_0)}}$ is bounded by an integrable function.

Remark 1: In the viability problems, one often assumes that the orientor field satisfies the following growth condition:

$$\|F(t,x)\| \le c(t)(1 + \|x\|) \tag{4}$$

where c is an L_1 function (see for instance [5], [9], [15]). This is a natural generalization of a similar growth condition assumed for differential equations to achieve existence of global solutions. However, the above example shows that for differential inclusions and viability problems this assumption is no longer needed.

Remark 2: In the above example the constraint multifunction K is not left absolutely continuous as it is required in viability theorem in [9]. Moreover, the orientor field F is not bounded by an integrable function on the entire graph of K, as it is required to be fulfilled in viability theorems in [5], [9], [15]. Therefore none of these theorems can be used to assert viability of the differential inclusion. But there exists a multiselection $K_{(t_0,y_0)}$ of K that is GDQ-differentiable at every point $t \in Do(K_{(t_0,y_0)})$ and $F \mid_{K_{(t_0,x_0)}}$ is bounded by an integrable function as we assume in our theorem. Thus there exists a viable solution for every $(t_0, x_0) \in Gr(K)$.

The measurable hypothesis H(F)(ii) in Theorem 8 seems to be as weak as it can. However, it is quite hard to verify this assumption. Thus we tried to formulate another measurable assumption in order to make simpler its verification. First we present the following proposition.

Proposition 3: Let $F : T \times \mathbb{R}^n \twoheadrightarrow \mathbb{R}^n$, $T = [0,a]$ and $Do(F) = T$, with compact convex values be measurable with respect to t and u.s.c. with respect to x. Then there exists a measurable map $y : [t_0, t_0 + \delta] \to \mathbb{R}^n$ such that $y(t) \in F(t, \gamma(t))$ for any $t \in [t_0, t_0 + \delta]$, where $\gamma : [t_0, t_0 + \delta] \to \mathbb{R}^n$ is any measurable map.

Proof: In order to show the thesis it is enough to prove that there exists a measurable multiselection $\tilde{F}$ of $F(t, \gamma(t))$. Let us take a measurable map $\gamma : [t_0, t_0 + \delta] \to \mathbb{R}^n$. Then there exists a sequence of simple maps $\{\gamma_k\}$ that converges pointwisely to γ with $k \to \infty$. Let us consider now the following multiselection $t \mapsto \text{Limsup}_{k\to\infty} F(t, \gamma(t))$. We show that for any t, $\text{Limsup}_{k\to\infty} F(t,\gamma(t)) \subseteq \text{Limsup}_{x' \to \gamma(t)} F(t, x')$. Let us take any $y \in \text{Limsup}_{k\to\infty} F(t, \gamma(t))$. Then

$$\forall\, \varepsilon > 0\ \forall\, N \ge 1\ \exists\, k(N) \ge N,\ y \in B(F(t, \gamma_k(t)), \varepsilon). \tag{5}$$

On the other hand, the convergence $\{\gamma_k\} \to \gamma$ implies that

$$\forall\, \eta > 0\ \exists\, N(\eta)\ \forall\, k \ge N,\ \gamma_k(t) \in B(\gamma(t), \eta). \tag{6}$$

Then for $k \ge N(\eta)$ we have from (5) and (6) that for every $\varepsilon > 0$ and for every $\eta > 0$ there exists $x' := \gamma_k(t)$ where $x' \in B(\gamma(t), \eta)$,

$$y \in B(F(t, x'), \varepsilon),$$

which means that $y \in \text{Limsup}_{x' \to \gamma(t)} F(t, x')$. Thus we get $\text{Limsup}_{k\to\infty} F(t, \gamma(t)) \subseteq \text{Limsup}_{x' \to \gamma(t)} F(t, x')$. By u.s.c. of F with respect to x, we have $\text{Limsup}_{x' \to \gamma(t)} F(t, x') = F(t, \gamma(t))$. Now, if we set $\tilde{F}(t) := \text{Limsup}_{k\to\infty} F(t, \gamma(t))$, it is left to be shown that this multifunction has nonempty closed values. The closedeness of $\tilde{F}$ arises directly from definition of Limsup. Since F has compact values and is u.s.c. with respect to x, $\text{Limsup}_{k\to\infty} F(t, \gamma(t)) = \bigcap_{N \ge 1} \overline{\bigcup_{k \ge N} F(t, \gamma_k(t))}$ is nonempty as a intersection of a sequence of descending compact sets. Therefore applying Kuratowski-Ryll-Nardzewski Theorem we get that there exists a measurable selection $y : [t_0, t_0 + \delta] \to \mathbb{R}^n$ such that $y(t) \in \tilde{F}(t)$ for any $t \in [t_0, t_0 + \delta]$. Now, since $\tilde{F}(t) \subseteq F(t, \gamma(t))$, we get $y(t) \in F(t, \gamma(t))$ for any $t \in [t_0, t_0 + \delta]$. ∎

Corollary 3: Let $F : G \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = G$ and G is a closed subset of $[0,a] \times \mathbb{R}^n$, be such that its extension is defined on $(G)^\varepsilon$ and preserves all properties of F for some ε. Then for such F Proposition 3 is valid.

Proof: It is enough to notice that for any measurable map γ there exists a sequence $\{\gamma_k\}$ of maps with countable number of values that uniformly converges to γ. Then the proof is similar to the proof of Proposition 3. ∎

IV. Conclusions and Future Work

We gave sufficient conditions for existence of a viable solution to a multivalued Cauchy problem with restrictions using generalized differential quotients as a differentiation tool in the tangential condition. Our future work is to find sufficient and maybe also necessary conditions guaranteeing that all solutions to a problem stay in the graph of a constraint multifunction K. This is the invariance problem for differential inclusions.

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