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Viability and generalized differential quotients [1]

by

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Abstract: Necessary and sufficient conditions for a set-valued map $K : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$ to be GDQ-differentiable are given. It is shown that K is GDQ differentiable at t_0 if and only if it has a local multiselection that is Cellina continuously approximable and Lipschitz at t_0. It is also shown that any minimal GDQ of K at (t_0, y_0) is a subset of the contingent derivative of K at (t_0, y_0), evaluated at 1. Then this fact is used to prove a viability theorem that asserts existence of a solution to the initial value problem $\dot{y}(t) \in F(t, y(t))$, with $y(t_0) = y_0$, where $F : Gr(K) \twoheadrightarrow \mathbb{R}^n$ is an orientor field (i.e. multivalued vector field) defined only on GrK and $K : T \twoheadrightarrow \mathbb{R}^n$ is a time-varying constraint multifunction. One of the assumptions is GDQ differentiability of K.

Keywords:
viability, differential inclusion, generalized differential quotient (GDQ), contingent derivative, Cellina continuous approximability (CCA).

1. Introduction

Viability problems are related to existence of global solutions of differential equations or differential inclusions whose dynamics are restricted to closed subsets of the state space. The first result in this area is due to Nagumo, who formulated necessary and sufficient conditions, under which all trajectories of a vector field starting at points of a closed set K_0 stay in this set. If we replace the differential equation by a differential inclusion, uniqueness of trajectories is lost and one may be interested in two different possibilities: either all trajectories starting from all points of K_0 stay in K_0 (forward invariance) or for each

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point of K_0 at least one trajectory starting at this point stays in K_0 (viability). Another extension of the problem leads to a differential inclusion with the right-hand side depending on time and the closed set changing in time as well.

Let $T = [0, b]$ and for each $t \in T$, let $K(t)$ be a subset of $\mathbb{R}^n$. Thus, K is a multifunction (set-valued map) whose graph GrK consists of points $(t, y) \in \mathbb{R} \times \mathbb{R}^n$ such that $t \in T$ and $y \in K(t)$. A time-dependent orientor field restricted by K is a multifunction F defined on GrK whose values are subsets of $\mathbb{R}^n$. Now, the multivalued Cauchy problem is defined as follows:

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{a.e. on } T \\ y(t_0) = y_0. \end{cases}$$

We are interested in conditions under which there exists $y : T \to \mathbb{R}^n$ that solves the Cauchy problem. As F is defined only on GrK, y has to satisfy the condition $y(t) \in K(t)$ for all $t \in T$.

There are many results concerning this problem. It seems that the theory achieved its maturity in the works of Bothe (1992), Frankowska, Plaskacz, Rzeżuchowski (1995) and Hu, Papageorgiou (1997) (see there for more references).

Our main result on viability follows Theorem 4.11 in Hu, Papageorgiou (1997b). But, instead of contingent derivatives used there (and also in many other sources) we exploit generalized differential quotients (GDQs) introduced recently by Sussmann (2000, 2002). We replace the assumption about existence of continuous half-selections of K made in Hu, Papageorgiou (1997b), which seems to be very technical, by a more natural assumption about GDQ-differentiablity of K. GDQ of K is also used instead of the contingent derivative in the main assumption needed for viability, which says that GDQ of K at (t, y) has a nonempty intersection with $F(t, y)$ for almost every $t \in T$ and all $y \in K(t)$.

Contrary to contingent derivatives, GDQs are not unique, so we are interested in minimal ones (in the sense of inclusion of sets). They contain the essential information on the (multi)function. SGDQ is the closure of the sum of all minimal GDQs. It appears as one of the main tools in the last section. Earlier, we show some properties of GDQs and study their connections with contingent derivatives. In most of the results we assume that multifunctions are defined on the real line, as we want to apply the theory to the restriction multifunction K that appears in the viability problem. In particular, we show that K is GDQ differentiable at t_0 if and only if it has a local multiselection that is Cellina continuously approximable and Lipschitz at t_0. The former property, denoted by CCA, was introduced by Cellina and recently rediscovered by Sussmann (in earlier papers under the name of 'regularity'). This concept is one of the key ingredients of the idea of GDQ and we present it in detail, showing some new facts. We also prove that any minimal GDQ of K at (t_0, y_0) is a subset of the contingent derivative of K at (t_0, y_0) evaluated at 1.

2. Basic notations and definitions

We follow, to some extent, the notation used in Sussmann (2002). By a set-valued map (multifunction) we mean a triple $F = (A, B, G)$ such that A and B are sets and G is a subset of $A \times B$. The sets A, B, G are, respectively, the *source*, *target* and *graph* of F, and we write $A = So(F)$, $B = Ta(F)$, $G = Gr(F)$. For $x \in So(F)$ we write $F(x) = \{y : (x, y) \in Gr(F)\}$ (it can happen that $F(x) = \emptyset$ for $x \in So(F)$). The sets $Do(F) = \{x \in So(F) : F(x) \neq \emptyset\}$, $Im(F) = \bigcup_{x \in So(F)} F(x)$, are, respectively, the *domain* and the *image* of F. If $F = (A, B, G)$ is a set-valued map, we say that F is a set-valued map from A to B, and write $F : A \twoheadrightarrow B$. We use $SVM(A, B)$ to denote the set of all set-valued maps from A to B. We reserve capital letters for set-valued maps and small ones for ordinary (single-valued and everywhere defined) maps.

If X is a metric space supplied with a distance d, $K \subseteq X$, then we denote the *distance from x to K* by $dist(x, K) := \inf_{y \in K} d(x, y)$, where we set $dist(x, \emptyset) := +\infty$. The *ball of radius $\epsilon > 0$ around K in X* is denoted by $B(K, \epsilon) := K^\epsilon := \{x \in X : dist(x, K) < \epsilon\}$. If $K = \{\bar{x}\}$, then we write $B(K, \epsilon) = B(\bar{x}, \epsilon)$. If X is also a linear space, the *unit ball*, denoted by B, is just $B(0, 1)$. The balls $B(K, \epsilon)$ are neighborhoods of K. When K is compact, each neighborhood of K contains such a ball around K.

Let X and Y be metric spaces. We say that a set-valued map $F : X \twoheadrightarrow Y$ is *upper semicontinuous* (abbreviated as u.s.c.) at $\bar{x} \in Do(F)$ if and only if for any neighborhood U of $F(\bar{x})$ there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$, $F(x) \subset U$. We say that F_n *graph converges* to F, and write $F_n \xrightarrow{gr} F$, if

$$lim_{n\to\infty} \triangle(Gr(F_n), Gr(F)) = 0$$

where

$$\triangle(A, B) = sup\{dist(q, B) : q \in A\}.$$

Let $\mathcal{T}$ be a metric space and $\{A_\tau\}_{\tau \in \mathcal{T}}$ be a family of subsets of a metric space X. The upper limit *Limsup* and the lower limit *Liminf* of A_τ at τ_0 are closed sets defined by

$$\text{Limsup}_{\tau\to\tau_0} A_\tau = \left\{ v \in X \,|\, \liminf_{\tau\to\tau_0} dist(v, A_\tau) = 0 \right\}$$

$$\text{Liminf}_{\tau\to\tau_0} A_\tau = \left\{ v \in X \,|\, \limsup_{\tau\to\tau_0} dist(v, A_\tau) = 0 \right\}.$$

A subset $A \subset X$ is said to be the limit of A_τ if

$$A = \text{Limsup}_{\tau\to\tau_0} = \text{Liminf}_{\tau\to\tau_0} =: \text{Lim}_{\tau\to\tau_0} A_\tau.$$

A set $C \subseteq \mathbb{R}^n$ is called a *cone* if $rx \in C$ for all $x \in C$ and $r \geq 0$. For $F \in SVM(\mathbb{R}^n, \mathbb{R}^m)$ we define $||F(x)|| := sup\{||y|| : y \in F(x)\}$ if $F(x) \neq \emptyset$ and set $||\emptyset|| = -\infty$.

3. Properties of CCA set-valued maps

DEFINITION 1 *Sussmann (2002) Let X and Y be metric spaces. A set-valued map $F : X \twoheadrightarrow Y$ is Cellina continuously approximable (abbreviated 'CCA') if for every compact subset K of X*
(1) $Gr(F\mid_K)$ is compact;
(2) there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of single-valued continuous maps $f_j : K \to Y$ such that $f_j \xrightarrow{gr} F\mid_K$.

We use $CCA(X,Y)$ to denote the set of all CCA set-valued maps from X to Y.

When $f : X \to Y$ is a single-valued map, then f belongs to $CCA(X,Y)$ if and only if f is continuous.

An important class of examples of CCA maps is provided by the following results.

THEOREM 1 *Sussmann (2002) Assume that K is a compact metric space, Y is a normed space, and C is a convex subset of Y. Let $F \in SVM(K,C)$ be a set-valued map such that the graph of F is compact and the value $F(x)$ is a nonempty convex set for every $x \in K$. Then F is CCA as a map from K to C.*

We can relax the assumption of compact graph by imposing compactness of the values of F and adding upper semicontinuity of F.

THEOREM 2 *Sussmann (2002) Assume that X is a metric space, Y is a normed space, and C is a convex subset of Y. Let $F \in SVM(X,C)$ be an upper semicontinuous set-valued map with nonempty compact convex values. Then $F \in CCA(X,C)$.*

THEOREM 3 *Sussmann (2002) Assume that X, Y, Z are metric spaces. Let $F \in SVM(X,Y)$, $G \in SVM(Y,Z)$. Then the composite map $G \circ F$ belongs to $CCA(X;Z)$.*

EXAMPLE 1 Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ such that

$$F(x) = \begin{cases} -1 & \text{if } x < 0 \\ [-1,1] & \text{if } x = 0. \\ 1 & \text{if } x > 0 \end{cases}$$

Then F is upper semicontinuous and has compact convex nonempty values. By Theorem 2 we conclude that F is CCA.

REMARK 1 Let $F : X \twoheadrightarrow Y$ be a CCA set-valued map. If A is a closed subset of X, then $F\mid_A$ is also a CCA set-valued map.

Definition 2 *Let X and Y be normed spaces. We say that a set-valued map $F : X \twoheadrightarrow Y$ is Lipschitz at the point x_0 if there exists $L \geq 0$ and a neighborhood $N(x_0)$ of x_0 such that*

$$\forall x \in N(x_0), \quad F(x) \subseteq F(x_0) + L||x - x_0||B$$

where B is a unit ball in Y.

Definition 3 *We say that a set-valued map $\tilde{F} : X \twoheadrightarrow Y$ is a multiselection of a set-valued map $F : X \twoheadrightarrow Y$ if for every $x \in X$, $\tilde{F}(x) \subseteq F(x)$.*

We need the following technical lemma to prove the next proposition.

Lemma 1 *Let U, U_1, U_2 be closed subsets of $\mathbb{R}^n$ such that $U = U_1 \cup U_2$. Let $F_1, F_2 : U \twoheadrightarrow \mathbb{R}^m$. Then for every $a > 0$*

$$\begin{aligned} &\triangle(Gr(F_1 |_{U_1}), Gr(F_2 |_{U_1})) < a \quad \text{and } \triangle(Gr(F_1 |_{U_2}), Gr(F_2 |_{U_2})) < a \\ &\qquad \Rightarrow \triangle(Gr(F_1), Gr(F_2)) < a. \end{aligned} \tag{1}$$

Proof. Assume that

$$\forall x \in U_1, \forall y \in F_1(x), \quad (x, y) \in (Gr(F_2 |_{U_1}))^a$$

and

$$\forall x \in U_2, \forall y \in F_1(x), \quad (x, y) \in (Gr(F_2 |_{U_2}))^a .$$

We know that $Gr(F_2) = Gr(F_2 |_{U_1}) \cup Gr(F_2 |_{U_2})$. Thus we have

$$(Gr(F_2 |_{U_1}))^a \subseteq (Gr(F_2))^a$$

and

$$(Gr(F_2 |_{U_2}))^a \subseteq (Gr(F_2))^a.$$

As any $(x, y) \in Gr(F_1)$ belongs either to $(Gr(F_2 |_{U_1}))^a$ or to $(Gr(F_2 |_{U_2}))^a$, we get that $(x, y) \in (Gr(F_2))^a$. ■

Proposition 1 *Let $U \subset \mathbb{R}$ be a compact neighborhood of 0 and $G : U \twoheadrightarrow \mathbb{R}^n$, $Do(G) = U$, be a set-valued map such that $G(0) = B\varrho$ and $G(x) \subseteq B\varrho$ for some $\varrho > 0$ and for $x \in U$. Let $G |_{U \setminus (-b,b)}$ be CCA for any $b > 0$. Then G is a CCA set-valued map on U.*

Proof. Let $G_n = G |_{U \setminus (-\frac{1}{n}, \frac{1}{n})}$, $n \in \mathbb{N}$. Since G_n is a CCA set-valued map, there exists a sequence $(f_{n,k})_{k \in \mathbb{N}}$, of continuous functions $f_{n,k} : U \setminus (-\frac{1}{n}, \frac{1}{n}) \to \mathbb{R}$, such that $\triangle(Gr(f_{n,k}), Gr(G_n)) \to 0$ when $k \to \infty$. It implies that

$$\forall n \ \exists \ k = k(n), \triangle(Gr(f_{n,k(n)}), Gr(G_n)) < \frac{1}{n} .$$

We can define a new sequence of continuous functions as follows

$$f_n(x) = \begin{cases} f_{n,k(n)}(x), & x \in U \backslash (-\frac{1}{n}, \frac{1}{n}) \\ \frac{n(x+\frac{1}{n}) f_{n,k(n)}(\frac{1}{n})}{2} - \frac{n(x-\frac{1}{n}) f_{n,k(n)}(-\frac{1}{n})}{2}, & x \in [-\frac{1}{n}, \frac{1}{n}] \,. \end{cases}$$

We will show that $\triangle(Gr(f_n), Gr(G))$ tends to 0 with $n \to \infty$. Since $f_n \equiv f_{n,k(n)}$ on the set $U \backslash \left(-\frac{1}{n}, \frac{1}{n}\right)$, we can write

$$\triangle \left(Gr \left(f_n \mid_{U \backslash (-\frac{1}{n}, \frac{1}{n})} \right), Gr \left(G \mid_{U \backslash (-\frac{1}{n}, \frac{1}{n})} \right) \right) < \frac{1}{n}. \tag{2}$$

Thus, in particular,

$$\left(\frac{1}{n}, f_n \left(\frac{1}{n} \right) \right) \in (Gr(G_n))^{\frac{1}{n}} \tag{3}$$

and

$$\left(-\frac{1}{n}, f_n \left(-\frac{1}{n} \right) \right) \in (Gr(G_n))^{\frac{1}{n}}. \tag{4}$$

Since f_n is a linear function on $[-\frac{1}{n}, \frac{1}{n}]$, $||f_n\left(\frac{1}{n}\right)||$ or $||f_n\left(-\frac{1}{n}\right)||$ is the maximal value of $||f_n||$ on $[-\frac{1}{n}, \frac{1}{n}]$.

We want to estimate the following distance

$$\triangle(Gr(f_n \mid_{[-\frac{1}{n}, \frac{1}{n}]}), Gr(G \mid_{[-\frac{1}{n}, \frac{1}{n}]})).$$

We know that for every $x \in U$, $G(x) \subseteq B\varrho$. Thus, $Gr(G) \subseteq U \times B\varrho$ and then

$$(Gr(G))^{\delta} \subseteq (U \times B\varrho)^{\delta} \tag{5}$$

for every $\delta > 0$. From (3), (4) and (5) it follows that

$$f_n \left(\pm \frac{1}{n} \right) \in (B\varrho)^{\frac{1}{n}} \,. \tag{6}$$

If $f_n(\frac{1}{n})$ and $f_n(-\frac{1}{n})$ are in $B\varrho$, then

$$\triangle \left(Gr \left(f_n \mid_{[-\frac{1}{n}, \frac{1}{n}]} \right), Gr \left(G \mid_{[-\frac{1}{n}, \frac{1}{n}]} \right) \right) < \frac{1}{n}.$$

Otherwise, from (6), $dist(f_n(\frac{1}{n}), B\varrho) < \frac{1}{n}$ and $dist(f_n(-\frac{1}{n}), B\varrho) < \frac{1}{n}$, so

$$\triangle \left(Gr \left(f_n \mid_{[-\frac{1}{n}, \frac{1}{n}]} \right), Gr \left(G \mid_{[-\frac{1}{n}, \frac{1}{n}]} \right) \right) < \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{\sqrt{2}}{n}.$$

Now we have to show that

$$\triangle(Gr(f_n), Gr(G)) \to 0$$

on the whole set U as $n \to \infty$. But this follows from Lemma 1, where we put $F_1 = f$, $F_2 = G$, $U_1 = \left[-\frac{1}{n}, \frac{1}{n}\right]$, $U_2 = U \setminus \left(-\frac{1}{n}, \frac{1}{n}\right)$. ■

Slightly changing the proof of Proposition 1 we get the following:

COROLLARY 1 *If G is defined on $[0,c]$, $G(0) = B\varrho$, $G(x) \subseteq B\varrho \subset \mathbb{R}^n$ for $x \in [0,c]$ and $G\mid_{[b,c]}$ is CCA for every $0 < b < c$, then G is CCA.*

The following lemma is used in Section 5 in order to prove an important relation between minimal GDQs and the contingent derivative.

LEMMA 2 *Let $G : [0,a] \twoheadrightarrow \mathbb{R}^n$ be a CCA set-valued map on $[0,a] \in \mathbb{R}$. Then the set-valued map $\tilde{G} : [0,a] \twoheadrightarrow \mathbb{R}^n$ defined as follows $Gr(\tilde{G}) := \overline{Gr(G)\mid_{(0,a]}}$ is CCA.*

Proof. First let us notice that the set $Gr(\tilde{G})$ is compact, because we take the closure of a bounded set $Gr(G)\mid_{(0,a]}$. This implies, in particular, that $\tilde{G}$ is u.s.c. at $x = 0$. Thus, we can write

$$\forall \varepsilon > 0 \ \exists\ \delta > 0 \ \forall x \in (0,\delta] : \ \tilde{G}(x) = G(x) \subseteq \tilde{G}(0) + \frac{\varepsilon}{2}B. \tag{7}$$

By Remark 1, $G\mid_{[\frac{\delta}{2},a]}$ is CCA. Thus, by putting $\delta' < \min\left\{\frac{\delta}{2}, \frac{\varepsilon}{2}\right\}$, we can find a continuous map $f : [\frac{\delta}{2}, a] \to \mathbb{R}^n$ such that $Gr(f) \subseteq \left(GrG\mid_{[\frac{\delta}{2},a]}\right)^{\delta'}$. This implies that for $x \in \left[\frac{\delta}{2}, \delta - \delta'\right]$ we get

$$f(x) \in (\tilde{G}(0))^{\frac{\varepsilon}{2}+\delta'}. \tag{8}$$

Indeed, we have

$$Gr(f) \subseteq \left(GrG\mid_{[\frac{\delta}{2},a]}\right)^{\delta'} \subseteq \left[\frac{\delta}{2}, \delta - \delta'\right] \times \tilde{G}(0)^{\frac{\varepsilon}{2}+\delta'} \cup [\delta - \delta', a] \times MB,$$

where $M := \max ||G(x)||$. Finally we can write

$$f(\frac{\delta}{2}) \in (\tilde{G}(0))^{\varepsilon}. \tag{9}$$

Then we can extend the map f to the interval $[0,\delta]$ in the following way

$$\tilde{f}(x) = \begin{cases} f(x), & x \in [\frac{\delta}{2}, a]; \\ f\left(\frac{\delta}{2}\right), & x \in [0, \frac{\delta}{2}]. \end{cases}$$

Hence, by (8) and (9), $Gr(\tilde{f}) \subseteq (Gr\tilde{G})^{\varepsilon}$. Therefore $\tilde{G}$ is CCA and the proof is finished. ■

4. GDQ-differentiability and minimal GDQs

Let us start with the definition of directional GDQ.

DEFINITION 4 *Sussmann (2002) Let* $m, n \in \mathbb{N}$, $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ *be a set-valued map,* $\overline{x} \in \mathbb{R}^m$, $\overline{y} \in \mathbb{R}^n$, $\overline{y} \in F(\overline{x})$ *and let* Λ *be a nonempty compact subset of* $\mathbb{R}^{n\times m}$ *(then an element of* Λ *is an* $n \times m$ *matrix). Let* S *be a subset of* $\mathbb{R}^m$*. We say that* Λ *is a generalized differential quotient (GDQ) of* F *at* $(\overline{x}, \overline{y})$ *in the direction of S, and write* $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ *if for every positive real number* δ *there exist* U, G *such that*
1. U *is a compact neighborhood of 0 in* $\mathbb{R}^m$ *and* $U \cap S$ *is compact;*
2. G *is a CCA set-valued map from* $\overline{x} + U \cap S$ *to the* δ*-neighborhood* Λ^δ *of* Λ *in* $\mathbb{R}^{n\times m}$*;*
3. $G(x) \cdot (x - \overline{x}) \subseteq F(x) - \overline{y}$ *for every* $x - \overline{x} \in U \cap S$*.*
For $S = \mathbb{R}^m$ *we write* $\Lambda \in GDQ(F; \overline{x}, \overline{y})$ *and say that* Λ *is a generalized differential quotient of* F *at* $(\overline{x}, \overline{y})$*.*

Observe that GDQs are not unique. If $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$, then for any compact overset Λ' of Λ also $\Lambda' \in GDQ(F; \overline{x}, \overline{y}; S)$.

We say that $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ is GDQ-differentiable at $(\overline{x}, \overline{y})$ in the direction of S if there exists at least one $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$.

DEFINITION 5 *Let* F *be GDQ-differentiable at* $(\overline{x}, \overline{y})$ *in the direction of* S*. A* minimal GDQ *of* F *at* $(\overline{x}, \overline{y})$ *in the direction* S *is a minimal element of the set of all GDQs of* F *at this point in the same direction (minimal in the sense of inclusions of sets).*

LEMMA 3 *Let* $F \in SVM(\mathbb{R}^n, \mathbb{R}^m)$ *and* $\Lambda_k \in GDQ(F; \overline{x}, \overline{y}; S)$ *for* $k \in \mathbb{N}$*. Let us assume that* $\Lambda_1 \supset \Lambda_2 \supset \ldots$ *and* $\bigcap_k \Lambda_k = \Lambda$*. Then*

$$\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)\,.$$

Proof. One can find the proof of this lemma in Girejko (2005). ■

THEOREM 4 (Minimality Theorem) *If the set of all GDQs of a set-valued map* F *at* $(\overline{x}, \overline{y})$ *in the direction of* S *is not empty, then there exists in this set at least one minimal GDQ at the same point and in the same direction.*

Proof. From the Kuratowski-Zorn Lemma, a family of sets with the property that descending sequences have a lower bound, possesses a minimal element. By Lemma 3, such a sequence (Λ_k) of GDQs has a lower bound – their intersection Λ. Thus in the family of all GDQs there exists a minimal element. ■

COROLLARY 2 *Every element* Λ *of* $GDQ(F; \overline{x}, \overline{y}; S)$ *contains a minimal element of* $GDQ(F; \overline{x}, \overline{y}; S)$ *in the sense of inclusion of sets.*

We use $minGDQ(F; x, y; S)$ to denote the collection of all minimal GDQs of F at (x, y) in the direction of S.

EXAMPLE 2 Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$. Then one can show that $[-1, 1] \in GDQ(f; 0, 0)$ and that this is the minimal GDQ. This interval is also the Clarke generalized gradient of f at 0. However, for $f(x) = x^2 \sin \frac{1}{x}$ when $x \neq 0$ and $f(0) = 0$ the same interval is again the Clarke generalized gradient of f at 0, while the minimal GDQ is just the ordinary derivative at 0, equal 0.

EXAMPLE 3 Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ be a set-valued map such that

$$F(x) = \begin{cases} [-|x|, |x|] & \text{if } x \neq 0 \\ \{0\} & \text{if } x = 0 \,. \end{cases}$$

Then any singleton $\{a\}$ for $a \in [-1, 1]$ is a minimal GDQ of F at $(0, 0)$.

THEOREM 5 *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$ be a set-valued map. Then, F is GDQ-differentiable at (x_0, y_0) if and only if there is a compact neighborhood U of x_0 such that $F\,|_U$ has a CCA multiselection $\tilde{F}$ that is Lipschitz at the point x_0 and $\tilde{F}(x_0) = y_0$.*

Proof. Without loss of generality, we can take $(x_0, y_0) = (0, 0)$. If we assume that F is GDQ-differentiable at $(0, 0)$, then directly from the definition of GDQ-differentiability it follows that F has a CCA and Lipschitz at 0 multiselection $\tilde{F}$ such that $\tilde{F}(0) = 0$. It is enough to take $\tilde{F}(x) := G(x) \cdot x$.

Assume now that F has a CCA multiselection $\tilde{F}$, Lipschitz at the point 0 and such that $\tilde{F}(0) = 0$. Then we have

$$\exists \tilde{U} \subset U \ \exists L > 0 \ \forall x \in \tilde{U}, \ \ \tilde{F}(x) \subseteq L \cdot B|x|$$

which implies that

$$\frac{\tilde{F}(x)}{x} \subseteq L \cdot B$$

for $x \neq 0$. Let us define

$$G(x) = \begin{cases} \frac{\tilde{F}(x)}{x}, & x \neq 0 \\ L \cdot B, & x = 0 \,. \end{cases} \tag{10}$$

We want to show that G defined in (10) is CCA. But this follows from Proposition 1, if we put $\tilde{G}(x) = \frac{\tilde{F}(x)}{x}$, $L = \varrho$ and show that $\frac{\tilde{F}(x)}{x}$ is CCA on $U \backslash (-b, b)$ for any $b > 0$. Indeed, we can write

$$\tilde{F}(x) \frac{1}{x} = M(\tilde{F}(x), \frac{1}{x}) = (M \circ (\tilde{F}, f))(x)$$

where $M(z,x) = z \cdot x$, $M : \mathbb{R}^{n+1} \to \mathbb{R}^n$ for $z \in \mathbb{R}^n$, $x \in \mathbb{R}$ and $f(x) = \frac{1}{x}$. We have $(\tilde{F}, f) : \mathbb{R} \twoheadrightarrow \mathbb{R}^n \times \mathbb{R}$. In order to use a product of CCA maps $(\tilde{F} \times f)(x,y) = (\tilde{F}(x), f(y))$ we define a continuous map $P : \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ as follows $P(x) = (x,x)$. Then we can compose $(\tilde{F}, f) = (\tilde{F} \times f) \circ P$. Since the product and composition of CCA maps is a CCA map, we get that $\frac{\tilde{F}(x)}{x}$ is CCA. Therefore, from Proposition 1 we conclude that G is a CCA set-valued map on U. It is clear that $G(x)x \subseteq F(x)$ for every $x \in \tilde{U}$. Thus the proof is finished. ■

Corollary 3 *$F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$ is GDQ-differentiable at $(0,0)$ in the direction $S = [0, +\infty)$ iff there is $U = [0, c]$ such that $F\mid_U$ has a CCA multiselection, Lipschitz at 0 and equal 0 at 0.*

5. Relations between GDQs and the contingent derivative

First let us recall the definition of the contingent derivative.

Definition 6 *(Aubin, Frankowska, 1990) Let X and Y be normed spaces. A set-valued map $F : K \twoheadrightarrow Y$, where $K \subset X$ and $Do(F) = K$, has the* contingent derivative *$DF(x_0, y_0)$ at $x_0 \in K$ and $y_0 \in F(x_0)$ if $DF(x_0, y_0)$ is a set-valued map from X to Y whose graph is the* contingent cone *$T_{Gr(F)}(x_0, y_0)$ to the graph of F at (x_0, y_0). In other words,*

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow (u_0, v_0) \in T_{Gr(F)}(x_0, y_0),$$

where the contingent cone (the 'Bouligand cone') to C at x is defined by

$$T_C(x) = \left\{ w \in X : \liminf_{t \downarrow 0} \frac{dist(x + tw, C)}{t} = 0 \right\}.$$

Equivalently, we can write:

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow \liminf_{h \to 0^+ u \to u_0} \ dist \left(v_0, \frac{F(x_0 + hu) - y_0}{h} \right) = 0.$$

When F is a locally Lipschitz set-valued map, the definition of the contingent derivative reduces to the following (see, e.g., Aubin, Cellina, 1984)

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow \liminf_{h \to 0^+} \ dist \left(v_0, \frac{F(x_0 + hu_0) - y_0}{h} \right) = 0.$$

Theorem 6 *Let $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$, $Do(F) = T \subseteq \mathbb{R}$, be a set-valued map and $y \in F(t)$. Then*

$$\Lambda \in minGDQ(F; t, y; \mathbb{R}_+) \ \Rightarrow \ \Lambda \subseteq DF(t,y)(1).$$

Proof. Without loss of generality we can assume that $(t, y) = (0, 0)$. Let $\Lambda \in minGDQ(F; 0, 0; \mathbb{R}_+)$. Then for $\varepsilon = \frac{1}{n}$ there exist $\delta_n > 0$ and a CCA set-valued map $G_n : [0, \delta_n] \to \Lambda^{\frac{1}{n}}$ such that $G_n(t)t \subseteq F(t)$ for $t \in [0, \delta_n]$. Then $G_n(t) \subseteq \frac{F(t)}{t}$. Let us assume that $\delta_n \to 0$ with $n \to \infty$. Define $W_k := \overline{GrG_k \mid_{(0,\delta_k]}}$ and $V_n = \overline{\bigcup_{k \geq n} W_k}$. Hence, (V_n) is a descending sequence. Therefore, we get $\bigcap_n V_n = \{0\} \times \Omega$, for some $\Omega \subseteq \mathbb{R}^n$. Let us notice that $\Omega \subseteq \Lambda$. Indeed, since for every $k \geq n$, $W_k \subseteq [0, \delta_n] \times \overline{\Lambda^{\frac{1}{n}}}$, thus $V_n \subseteq [0, \delta_n] \times \overline{\Lambda^{\frac{1}{n}}}$. This implies $\Omega \subseteq \Lambda$. We need now the following lemma.

Lemma 4 *Ω is a GDQ of F at the point $(0, 0)$ in the direction of $\mathbb{R}_+$.*

Proof. Let $\varepsilon > 0$. There is $k \in \mathbb{N}$ such that $V_k \subseteq (\{0\} \times \Omega)^{\frac{\varepsilon}{2}}$. Thus, also $W_k \subseteq (\{0\} \times \Omega)^{\frac{\varepsilon}{2}}$. Define $GrG := \overline{GrG_k \mid_{(0,\delta_k]}}$. By definition, $GrG = W_k$ and from Lemma 2, G is CCA. Moreover, $G(0) \subseteq \Omega^{\frac{\varepsilon}{2}}$, so from u.s.c. of G at 0,

$$\exists\ \eta_k\ \forall\ t \in [0, \eta_k],\ G(t) \subseteq G(0)^{\frac{\varepsilon}{2}} \subseteq \Omega^{\varepsilon}.$$

As $G_k(t)t \subseteq F(t)$ for every $k > 0$ and $t \in [0, \delta_k]$, then also $G(t)t \subseteq F(t)$. Thus G restricted to $[0, \eta_k]$ is the required CCA set-valued map, so $\Omega \in GDQ(F; 0, 0; \mathbb{R}_+)$. ■

Since Ω is a GDQ and $\Omega \subseteq \Lambda$, we get $\Omega = \Lambda$. Let $v \in \Omega$. Thus, there exists a sequence $\{v_k\}$ converging to v such that $v_k \in G_k(t_k) \subseteq \frac{F(t_k)}{t_k}$ for $k \geq n$ and some $t_k > 0$. This implies that $\lim_{k\to\infty} dist(v, \frac{F(t_k)}{t_k}) = 0$. Therefore, $v \in DF(0, 0)(1)$ and the proof is complete. ■

Remark 2 Similarly, one can show that

$$\Lambda \in minGDQ(F; t, y; \mathbb{R}_-)\ \Rightarrow\ \Lambda \subseteq DF(t, y)(-1).$$

Remark 3 If Λ is not a minimal *GDQ*, then for a fixed family of maps G_k defining Λ, the set Ω constructed in the proof of Theorem 6 is an ‘optimal’ GDQ without the possibly superfluous points of the set Λ.

Corollary 4 *Consider $F : \mathbb{R} \twoheadrightarrow \mathbb{R}^n$. If F is GDQ-differentiable at the point (x, y) in the direction of $\mathbb{R}_+$ $(\mathbb{R}_-)$, then there exists the contingent derivative $DF(t, y)(1)$ $(DF(t, y)(-1))$.*

Proof. Since F is GDQ-differentiable at (t, y) in the direction $\mathbb{R}_+$ $(\mathbb{R}_-)$, there exists at least one minimal GDQ of F at this point, which is contained in $DF(t, y)(1)$ $(DF(t, y)(-1))$. ■

6. Viability results

Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T = [0, b] \subseteq \mathbb{R}$, be a constraint multifunction and $F : GrK \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = GrK$, be an orientor field (i.e. multivalued vector field). Consider the multivalued Cauchy problem as follows:

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{a.e. on } T \\ y(t_0) = y_0 \in K(t_0). \end{cases} \tag{11}$$

By a solution of this problem we mean an absolutely continuous function $y : [t_0, b] \to \mathbb{R}^n$ that satisfies the inclusion almost everywhere and satisfies the initial condition, with $y(t) \in K(t)$ for $t \in [t_0, b]$.

Let $SGDQ(K; t, y; \mathbb{R}_+)$ denote the closure of the union of all minimal GDQs of K at $(t, y) \in GrK$ in the direction of $\mathbb{R}_+$

We say that $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T$, is *left u.s.c.* if to every $t_0 \in (0, a]$ and $\varepsilon > 0$ there is a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$K(t) \subseteq K(t_0) + \varepsilon B(0)$$

for all $t \in (t_0 - \delta, t_0] \cap T$.

The following proposition shows the existence of a solution to (11), under GDQ-differentiability condition on K.

Proposition 2 *Let $K : T \twoheadrightarrow \mathbb{R}^n$ be GDQ-differentiable at every $(t, y) \in GrK$ in the direction of $\mathbb{R}_+$. Assume that K is left u.s.c. with closed values. Let $F : GrK \twoheadrightarrow \mathbb{R}^n$, with $Do(F) = GrK$, be a multifunction with closed convex values such that*

1. *for every $\varepsilon > 0$ there is a closed set $T_\varepsilon \subseteq T$ with $\lambda(T \backslash T_\varepsilon) \leq \varepsilon$ such that $F \mid_{(T_\varepsilon \times \mathbb{R}^n) \cap Gr(K)}$ is u.s.c.;*
2. *there exists $N \subset [0, b]$, $\lambda(N) = 0$, such that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ for all $(t, y) \in ((T \backslash N) \times \mathbb{R}^n) \cap Gr(K)$, and*
3. *$|F(t, y)| \leq a(t)(1 + ||y||)$ for almost every $t \in T$ and for every $y \in K(t)$, with $a \in L^1(T)$,*

where λ is the Lebesgue measure. Then (11) has a solution $y : [t_0, b] \to \mathbb{R}^n$, which is an absolutely continuous function.

Proof. By Theorem 6, if $\Lambda \in minGDQ(K; t, y; \mathbb{R}_+)$ then $\Lambda \subseteq DK(t, y)(1)$. Thus the conclusion of the proposition follows from Theorem 1 in Bothe (1992) in which $DK(t, y)(1)$ is used instead of $SGDQ(K; t, y; \mathbb{R}_+)$. ■

Proposition 3 *Assume that $G \subseteq \mathbb{R} \times \mathbb{R}^n$, $F : G \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = G$, and that for almost all t the map $F(t, \cdot)$ is upper semicontinuous with compact convex values. Then there exists a multifunction $F_0 : G \twoheadrightarrow \mathbb{R}^n$ with compact convex values such that $F_0(t, y) \subseteq F(t, y)$ for $(t, y) \in G$, and satisfying the conditions*

1. *for every $\varepsilon > 0$ there exists a measurable set $T_\varepsilon \subset \mathbb{R}$ such that $\lambda(\mathbb{R} \backslash T_\varepsilon) < \varepsilon$ and the multifunction $F_0 \mid_{(T_\varepsilon \times \mathbb{R}^n) \cap G}$ is u.s.c.;*

2. *if $T \subset \mathbb{R}$ is measurable , $u : T \to \mathbb{R}^n$ and $v : T \to \mathbb{R}^n$ are measurable maps such that $v(t) \in F(t, u(t))$, for almost all $t \in T$, then $v(t) \in F_0(t, u(t))$ for almost all $t \in T$.*

Proof. For the proof see Jarnik, Kurzweil (1968). ■

PROPOSITION 4 *If K is GDQ-differentiable at (s, y) in the direction $\mathbb{R}_+$, then there exists a measurable map $\gamma : [s, s+\delta] \to \mathbb{R}^n$ such that $\gamma(t) \in K(t)$ for $t \in [s, s+\delta]$, $\gamma(s) = y$ and γ is continuous at s.*

Proof. Without loss of generality, we can assume $s = 0$, $y = 0$. From GDQ-differentiability of K, there is a CCA set-valued map G such that $G(t)t \subseteq K(t)$ for $t \in [0, \delta]$. Since G has compact graph, then, in particular, it is u.s.c. and has closed values, which implies that $H(t) := G(t)t$ is also u.s.c. and has closed values. Then, by Proposition 8.2.1 and Theorem 8.1.3 in Aubin, Frankowska (1990) we get that there exists a measurable map $\gamma : [0, \delta] \to \mathbb{R}^n$ such that $\gamma(t) \in H(t) \subset K(t)$. Moreover, if $M := \max_{t\in[s,s+\delta]} ||G(t)||$, then $||\gamma(t)|| \leq Mt \to 0$ with $t \to 0$, and the proof is complete. ■

Now we present the main result of this paper.

THEOREM 7 *Consider a multivalued Cauchy problem (11). Assume that $K : T \twoheadrightarrow \mathbb{R}^n$, where $T = [0, b]$, is a left u.s.c. multifunction with nonempty closed values such that for all $(t, y) \in GrK$, where $t \in [0, b)$, K is GDQ differentiable at (t, y) in the direction of $\mathbb{R}_+$, and for every $\varepsilon > 0$ there exists $S_\varepsilon \subseteq T$ such that $\lambda(T \backslash S_\varepsilon) < \varepsilon$ and the map $(t, y) \mapsto SGDQ(K; t, y; \mathbb{R}_+)$ is u.s.c. on $(S_\varepsilon \times \mathbb{R}^n) \cap GrK$. Let $F : GrK \twoheadrightarrow \mathbb{R}^n$ with nonempty closed convex values satisfy*
(a) for every measurable $\gamma(\cdot)$ the map $t \mapsto F(t, \gamma(t))$ is measurable;
(b) the map $y \mapsto F(t, y)$ is u.s.c. for every $t \in [0, b]$;
(c) $|F(t, y)| \leq a(t) + c(t)||y||$ a.e. on T with $a, c \in L^1(T)$.
Additionally, assume that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ for almost every t, $(t, y) \in GrK$. Then for every $y_0 \in K(t_0)$, the problem (11) has an absolutely continuous solution.

Proof. The idea of the proof comes from the proof of Theorem 4.11 in Hu, Papageorgiou (1997b). First, let us notice that by assumptions (a),(b),(c) on the set-valued map F and from Proposition 3 we get that there exists $\tilde{F} : GrK \twoheadrightarrow \mathbb{R}^n$ with compact convex values such that
(i) $\tilde{F}(t, y) \subseteq F(t, y)$ for all $(t, y) \in GrK$;
(ii) for every $\varepsilon > 0$ there exists a closed set $T_\varepsilon \subseteq T$ such that $\lambda(T \backslash T_\varepsilon) < \varepsilon$ and $\tilde{F}\,|_{(T_\varepsilon \times \mathbb{R}^n) \cap GrK}$ is u.s.c.; and
(iii) if $A \subseteq T$ is measurable and $\alpha, \beta : A \to \mathbb{R}^n$ are measurable functions such that $\beta(t) \in F(t, \alpha(t))$ a.e. on A, then $\beta(t) \in \tilde{F}(t, \alpha(t))$ a.e. on A.
By Proposition 2, it is enough to prove now that for almost all $t \in T$ and every $y \in K(t)$, we have $\tilde{F}(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$. In order to prove this, we start with $\varepsilon > 0$. Let S_ε be a subset of T from the assumption on K. Let

$T_\varepsilon \subseteq T$ be a set satisfying the property (ii) above. From Lebesgue's density theorem, almost all points of $T_\varepsilon \cap S_\varepsilon$ are density points. Therefore, there exists a set $\tilde{T}_\varepsilon \subseteq T_\varepsilon \cap S_\varepsilon$ with $\lambda(T \setminus \tilde{T}_\varepsilon) \leq 3\varepsilon$ such that the points in $\tilde{T}_\varepsilon$ are right density points of $T_\varepsilon \cap S_\varepsilon$. Hence, for each $s \in \tilde{T}_\varepsilon$ there is (t_n) such that $t_n \in T_\varepsilon \cap S_\varepsilon$, $t_n > s$ and $t_n \to s$. Let us assume that $b \notin \tilde{T}_\varepsilon$. Let $s \in \tilde{T}_\varepsilon$ and $y \in K(s)$. From GDQ-differentiability of K at (s, y) in the direction $\mathbb{R}_+$ and Proposition 4, we get that there exists on $[s, s+\rho]$ a single-valued map $t \to \gamma(t) \in K(t)$, measurable and continuous at s, with $\gamma(s) = y$. Because of the assumption (a) on F, the map $t \to F(t, \gamma(t))$ is measurable on $[s, s+\rho]$. From the almost u.s.c. assumption on the map $(t, y) \mapsto SGDQ(K; t, y; \mathbb{R}_+)$, by Lemma 2 in Rzeżuchowski (1980), there is a subset M of T such that $\lambda(T \setminus M) = 0$ and the restriction of this map to $(M \times \mathbb{R}^n) \cap Gr(K)$ is jointly measurable. Then the map $t \mapsto SGDQ(K; t, \gamma(t); \mathbb{R}_+)$ is measurable on $M \cap [s, s+\rho]$ and thus on $[s, s+\rho]$, and so is $t \mapsto F(t, \gamma(t)) \cap SGDQ(K; t, \gamma(t); \mathbb{R}_+)$. Moreover, from the assumption that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ for a.e. $t \in [s, s+\rho]$, it has nonempty values for a.e. $t \in [s, s+\rho]$. Now, applying Proposition on measurable selection from Rzeżuchowski (1980), which says that a measurable set-valued map with (possibly empty) closed values has a measurable selection, we get a measurable function $t \to y(t)$ such that

$$y(t) \in F(t, \gamma(t)) \cap SGDQ(K; t, \gamma(t); \mathbb{R}_+) \tag{12}$$

for a.e. $t \in [s, s+\rho]$. By property (iii) of $\tilde{F}$, we can write $y(t) \in \tilde{F}(t, \gamma(t)) \cap SGDQ(K; t, \gamma(t); \mathbb{R}_+)$ for a.e. $t \in [s, s+\rho]$. The fact that s is a right density point of $T_\varepsilon \cap S_\varepsilon$ implies that there exists $t_n \to s$, $t_n > s$ such that $\{t_n\}_{n \geq 1} \subseteq T_\varepsilon \cap S_\varepsilon \cap S$ where $S = [s, s+\rho] \setminus N$, $\lambda(N) = 0$, and

$$y_n := y(t_n) \in \tilde{F}(t_n, \gamma(t_n)) \cap SGDQ(K; t_n, \gamma(t_n); \mathbb{R}_+)$$

where $y(t_n)$ is given by (12). Since $\tilde{F} \mid_{(T_\varepsilon \times \mathbb{R}^n) \cap GrK}$ is u.s.c. with compact values, thus $\overline{\bigcup_{n \geq 1} \tilde{F}(t_n, \gamma(t_n))}$ is compact. Therefore we may assume that $y_n \to \tilde{y}$ as $n \to \infty$ and since $\mathrm{Limsup}_{t_n \to s} \tilde{F}(t_n, \gamma(t_n)) \subseteq \tilde{F}(s, y)$ then $\tilde{y} \in \tilde{F}(s, y)$. Analogously, from u.s.c. of SGDQ of K, $\tilde{y} \in SGDQ(K; s, y; \mathbb{R}_+)$, so $\tilde{F}(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$, for a.e. $t \in T$ and every $y \in K(t)$. Then Proposition 2 permits to conclude. ■

Remark 4 In the above theorem the assumption on K to be GDQ-differentiable at every $(t, y) \in (T \times \mathbb{R}^n) \cap GrK$ is important. Indeed, let $T = [0, 1]$ and $y : T \to \mathbb{R}$ be the Cantor function. Thus y is continuous, nondecreasing, $\dot{y}(t) = 0$ for almost every $t \in T$, $y(T) = T$ and $y(\cdot)$ is not absolutely continuous. Let $K(t) = \{y(t)\}$ and $F(t, y) = \{0\}$. Then the tangential condition is satisfied for all $t \in T \setminus N$, $\lambda(N) = 0$, such that $\dot{y}(t) = 0$. However, problem (11) has no solution since $0 \notin K(t)$ for $t > 0$.

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