Systems on universe spaces*

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Abstract

The paper studies the class of analytic input/state/output systems for which the vector fields and the observation functions are only partially defined. The state space of the system is a universe space - a newly defined concept. As an example, quotient systems and observability are studied in detail. Observability is defined in a new way, consistent with the "universal" approach.

Key words. Function universe. Universe space. Partially defined function. Observability. Quotient system. Vector field.

1 Introduction

The aim of this paper is to develop a new concept of nonlinear control system (cf. Section 2), one that lives on a new type of state space which we call a universe space. Universe spaces are far-reaching generalizations of analytic manifolds. There are two reasons for introducing systems based on this new type of state space. The first comes from natural limitations of the class of analytic manifolds, which often serve as state spaces for nonlinear systems. For instance, quotient spaces of analytic manifolds generally are not analytic manifolds, and they don't even need to be analytic spaces. Moreover construction of quotient spaces can be difficult even in those cases when it is possible. This is an obstacle when we want to construct an observable system from one which is not. In [B3] algebraic varieties, which are not necessarily manifolds, served as state spaces, but one could not consider ideas which were analytic and not algebraic. Varieties as state spaces had been used earlier in [S1] for discrete-time systems.

The second reason for this new type of state space is that universe spaces allow the data of the control system (i.e. vector fields, constituting the dynamics, and output functions) to be only partially defined on the state space. This feature, almost entirely neglected in the literature on systems theory, is interesting

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and important from both the theoretical and practical points of view. In our broader class of systems we can include rational systems, where vector fields and output functions might not be defined at certain points of the state space (see [B2] for details). Moreover, we shall see in Section 6 that systems with globally defined output functions can lead naturally to systems with outputs that are not globally defined. In any case, partially defined functions seem endemic to the concerns of control theory, because a measuring device might not be able to access all parts of a system, or might fail to operate at certain points of it.

The classical indistinguishability relation defined by a nonlinear system (see Section 2) might not be an equivalence relation. If the vector fields of the system are not complete (or the system is partially defined) transitivity can fail, and this is one of the reasons why authors avoid incomplete systems (e.g. [HK]). However this drawback can be removed by changing slightly the definition of indistinguishability. We shall add one more chance to distinguish two states. This will be the case when, due to incompleteness, one point produces blowup of the trajectory and the other does not. This possibility has obvious theoretical appeal even though there are cases where one would not wish to perform such experiments. We shall generalize this case to partially defined systems by associating to a blowup of the trajectory a blowup of all the outputs.

The conventional approach can easily become burdened by the necessity of remembering domains of definition, if we were to expand it so as to allow for partially defined entities. The approach which we advocate deals effectively with that problem, for it handles partially defined functions and vector fields without any need for frequent and encumbering reference to the region where our statements hold. This is achieved by concentrating on the formal algebraic properties of partially defined functions rather than on the values they assume at the various points of their domains. Since the values functions take on are easily understood in terms of their formal algebraic properties, this emphasis does not put them out of our reach.

The idea of universe space follows the theory of universes developed by Johnson in [J1]. Universes are flexible tools that sew together local and global ideas. In algebras, only addition and multiplication are available; a universe offers infinitely many C^ω (or C^∞) operations. For instance, if M is a C^ω manifold, $\varphi_1, \ldots, \varphi_k$ globally defined C^ω functions on M and F a globally defined C^ω function on $\mathbb{R}^k$, we have the globally defined C^ω function $F \circ (\varphi_1, \ldots, \varphi_k)$ on M. But if $\varphi_1, \ldots, \varphi_k$ and F are just partially defined, $F \circ (\varphi_1, \ldots, \varphi_k)$ can denote a partially defined C^ω function on M, defined similarly, but with open (perhaps empty) domain consisting of exactly those x in M for which $F(\varphi_1(x), \ldots, \varphi_k(x))$ is meaningful. So if $\mathcal{O}_M$ denotes the set of all partially defined C^ω functions on open sets of M, the operations

$$(\varphi_1, \ldots, \varphi_k) \mapsto F \circ (\varphi_1, \ldots, \varphi_k)$$

are defined for all $\varphi_i \in \mathcal{O}_M$, $F \in \mathcal{O}_{\mathbb{R}^k}$ and $k > 0$. And $\mathcal{O}_M$ together with these operations is a universe, or more specifically – a function universe. An important point is that everything about M can be encoded in $\mathcal{O}_M$ using only these operations and their formal properties. When we study control theory here, these substitutions will play the primary role.

We introduce vector fields on universe spaces and study their trajectories. With reasonable assumptions one finds that trajectories are locally unique. Existence is a more complicated matter, but local existence of trajectories tends not to present a problem in many examples. Formation of quotient objects is very natural using the category of universe spaces and can be carried out without any special restrictions as long as we use a subuniverse to define the equivalence relation. This allows for construction of observable systems from unobservable ones, so we can apply our theory to that problem. This is in the spirit of [B1], where ringed spaces were used instead of universe spaces. There is a bit of a sheaf-theoretic flavor in our approach, but in fact the methods we use here are rather algebraic in nature (see the end of Section 3 for more comments). Besides providing examples for the theory throughout, we placed many at the end. The reader may want to consult from the start those placed there, even though they may exploit ideas introduced later in the text.

Modulo the precision provided by our definitions, we can state here some of the results we have obtained for our broader notion of system. Among them are the following :

- Indistinguishable states are also infinitesimally indistinguishable (Theorem 6.6). Under appropriate conditions, this implication can be reversed (Theorem 7.1).
- Quotient systems exist (Proposition 7.6).
- For a given unobservable system we construct an observable one with the same set of response maps (Theorem 8.5).

For the convenience of the readers who are not familiar with the standard theory of nonlinear systems, we provide basic definitions and results we shall need in Section 2. Section 3 is a short introduction to function universes and Section 4 gives the definition of universe space and its basic properties. In Section 5 we define vector fields on universe spaces and study existence and uniqueness of their trajectories. Systems on universe spaces are introduced in Section 6. Section 7 shows how to construct quotient systems in the class of systems on universe spaces. This is used in Section 8 for passing from unobservable systems to observable ones. Section 9 provides a number of interesting examples and Section 10 gives a perspective for future work.

2 Short introduction to nonlinear systems

This section is intended mainly for readers unfamiliar with nonlinear control systems (more details may be found in [I],[HK],[S2]). The experts will find here some motivation for our approach.

Let us consider the following system of equations:

$$\dot{x} = f(x,u) \tag{1}$$
$$y = h(x). \tag{2}$$

By $\dot{x}$ we denote the derivative of x with respect to the time variable t. We assume that for every $t \in \mathrm{I\!R}$, $x(t)$ belongs to a C^∞ or C^ω manifold M which is called the *state space*. The right hand side of (1) depends on the *state* x and a *control (function)* $u : [0,T_u] \to \Omega$. Sometimes it is assumed that the *set of control values* Ω is contained in some $\mathrm{I\!R}^m$, but for our purposes it may be an arbitrary set. For many problems it is enough to assume that controls u are piecewise constant. For any fixed value $\omega \in \Omega$, $f(\cdot,\omega)$ is a vector field on M. Thus (1), called the *dynamics*, may be represented by the family of vector fields $\{f_\omega, \omega \in \Omega\}$, where $f_\omega = f(\cdot,\omega)$. With any control $u : [0,T_u] \to \Omega$ and any initial condition $x_0 \in M$ we may associate the trajectory of (1) which is obtained by concatenating the trajectories of vector fields corresponding to subsequent values of u. This gives a piecewise C^∞ or C^ω curve on M. The trajectory may not be defined on the entire interval $[0,T_u]$; it may blow up earlier if vector fields defining the dynamics are not complete.

Equation (2) represents the *observation structure* and the map $h : M \to \mathrm{I\!R}^r$ is called the observation or output map. This map models a device which is used to observe (a part of) the internal dynamics given by x. Equations (1) and (2) define a *control system with output* or simply a *system*, usually denoted by Σ.

Let $\gamma(t,x_0,u)$ denote the trajectory of Σ corresponding to x_0 and u, and evaluated at time t. If p and q are two points in M, then we call them *classically indistinguishable* if

$$h(\gamma(t,p,u)) = h(\gamma(t,q,u)), \tag{3}$$

for every control u and for every $t \geq 0$ for which both sides of (3) are defined. We say that Σ is *classically observable* if it does not have classically indistinguishable distinct points.

Let $p \in M$ and let U_p denote the *set of admissible controls*, i.e. the controls u for which $\gamma(t,p,u)$ is defined on the same interval as u. By the *(classical) response map* at p we mean a map $S_p : U_p \to \mathrm{I\!R}^r$ defined by

$$S_p(u) = h(\gamma(T_u,p,u)),$$

where T_u is the time for which the control u is defined. In Section 6 we extend this definition to partially defined systems with possibly infinitely many output functions. Observe that classical indistinguishability of p and q means that $S_p = S_q$ on $U_p \cap U_q$.

Classical indistinguishability is not, in general, an equivalence relation. However, it *is* one in two important cases: when Σ is analytic and when Σ is C^∞ but vector fields are complete, i.e. trajectories do not blow up. In these cases one may try to form a quotient state space gluing up classically indistinguishable states. The quotient space is always well defined on the topological level, but it may not be a differential manifold.

Let $\mathcal{L}(\Sigma)$ be the Lie algebra over $\mathbb{R}$ generated by the vector fields $f_\omega, \omega \in \Omega$. We say that Σ is *transitive* if $\mathcal{L}_x(\Sigma) = T_x M$ for every $x \in M$, i.e. the Lie algebra of system Σ spans the whole tangent space at each point. For analytic systems transitivity is equivalent to accessibility which means that the reachable set from any point has a nonempty interior (see [S2],[HK]).

Theorem 2.1 (Sussmann) *Assume that Σ is analytic or has complete vector fields. If Σ is transitive then the quotient space with respect to the classical indistinguishability relation is a differential manifold.* ♠.

One of our goals in this paper is to examine what happens if we drop all the assumptions in the above theorem. Moreover we want our system to be only partially defined. This means that vector fields f_ω and components h_i of h are defined on open subsets of M, depending on ω and i. For such a system the classical indistinguishability relation may fail to be transitive even in the analytic case.

Example 2.2 *Let $M = \mathbb{R}^2$ and let the dynamics consist of only one everywhere defined vector field $f(x_1, x_2) = (1, 0)$ (identified with the first order differential operator $\partial_1 = \partial/\partial x_1$). Let $r = 2$, $h_1(x_1, x_2) = x_1$ for all $x \in M$, and $h_2(x_1, x_2) = x_2$ be defined only for points x with $|x_2| < 2$ (imagine a measuring device with these constraints). Consider points $m = (0, 3)$, $p = (0, 1)$ and $q = (0, -1)$. The points m and p are classically indistinguishable, since only h_1 may be used for distinguishing purposes (h_2 not defined at m). The same holds for points m and q. However, we can easily distinguish p and q, because h_2 may be applied to both of them. Thus the classical indistinguishability is not transitive in this case.*♠

The above example clearly indicates how to improve the definition. Namely, if one of the sides of (3) is not defined for some t while the other side is defined for that t, we should classify the points as distinguishable. For globally defined systems this may happen because of blowup of one of the trajectories, a very good reason to distinguish points. In the case of partially defined systems, such a feature may be due to the fact that one of the points belongs to the domain of a vector field or an output map, while the other does not. To our knowledge such an extended definition never appeared in the literature. It fits naturally in the universe spaces framework and may be simply expressed and manipulated with the language introduced in next sections.

Another non-classical approach to observability, based on the differential algebraic methods, has been developed in [DF].

3 Function universes

Details for this section are in [J1]. Below we give only necessary definitions and statements, leaving aside some other important aspects of universes that could be used in control theory.

Let X be a set. A *partially defined function* on X is any map $\varphi : A \to \mathbb{R}$ where A is a subset of X, called the *domain* of φ and written dom φ. Let $\mathbb{R}_X$ denote the set of partially defined functions on X. It is useful to extend any $\varphi \in \mathbb{R}_X$ to the entire set X by assigning $\varphi(x) = \varphi x = \emptyset_0$ for $x \notin \text{dom } \varphi$. We call $\emptyset_0$ the *phantom.* It is an extra point added to the real line that symbolizes the idea of "undefined scalar". However, the word *scalar* will never refer to $\emptyset_0$; it will just mean an arbitrary element of $\mathbb{R}$. We let $\mathbb{A}_0 = \mathbb{R} \cup \{\emptyset_0\}$ and we identify any φ in $\mathbb{R}_X$ with its extension $\varphi : X \to \mathbb{A}_0$ that we have defined, so dom $\varphi = \{x \in X : \varphi x \neq \emptyset_0\}$. Thus $\mathbb{R}_X$ consists of all maps $X \to \mathbb{A}_0$. We note special elements $a_X \in \mathbb{R}_X$ defined for $a \in \mathbb{R}$ by $a_X(x) = a,\ x \in X$, and also $\emptyset_X \in \mathbb{R}_X$ defined by $\emptyset_X(x) = \emptyset_0,\ x \in X$. We shall use 0_X extensively throughout the paper.

If $f : X \to Y$ is a map of sets, we have a map $f^* : \mathbb{R}_Y \to \mathbb{R}_X$ defined by

$$(f^*\varphi)(x) = \varphi(fx),\ \varphi \in \mathbb{R}_Y,\ x \in X,$$

i.e. $f^*\varphi = \varphi \circ f = \varphi f$. (We often drop parenthesis and $\circ$ when this does not lead to confusion.) Evidently $f^* a_Y = a_X,\ a \in \mathbb{R}$, and $f^*\emptyset_Y = \emptyset_X$. We call f^* the transposition of f.

Let $\mathbb{R}^0$ denote a one-point set $\{*\}$. We have a natural identification $\mathbb{R}_{\mathbb{R}^0} \simeq \mathbb{A}_0$ given by $\varphi \mapsto \varphi(*)$. It identifies $\emptyset_{\mathbb{R}^0}$ with $\emptyset_0$. If X is any C^r manifold, let $\mathcal{O}_X = \{\varphi \in \mathbb{R}_X : \text{dom } \varphi \text{ open},\ \varphi_{|\text{dom}\varphi} \text{ is } C^r\}$.

A subset U of $\mathbb{A}_0$ is *open* if $U = \mathbb{A}_0$ or if U is an open subset of $\mathbb{R}$. This defines a topology on $\mathbb{A}_0$ such that if X is a C^ω manifold, every φ in $\mathcal{O}_X$ maps X continuously into $\mathbb{A}_0$. Also our topology on X is the one with the fewest open sets for which this is so. We note that $\mathbb{R}$ is a Hausdorff subspace of $\mathbb{A}_0$ and that $\{\lambda, \emptyset_0\}$ is the closure of $\{\lambda\}$ for any λ in $\mathbb{A}_0$.

We shall concentrate on the real-analytic theory. There are analogous C^∞ and complex-analytic theories as well, but our methods will not always apply to the C^∞ case, which sometimes has a very different flavor. We do provide, however, a C^∞ example (Example 7.9) to show that the theory can be applied to C^∞ systems as well.

If the manifold X considered above is $\mathbb{R}^n$, $n > 0$, we get $\mathcal{O}_{\mathbb{R}^n}$, which we denote by $\mathbb{A}_n$. Any element $F : \mathbb{R}^n \to \mathbb{A}_0$ of $\mathbb{A}_n$ is a partially defined analytic function of n variables with open domain. It has a natural extension to a map $F : (\mathbb{A}_0)^n \to \mathbb{A}_0$, also denoted F, with $F(a_1, \ldots, a_n) = \emptyset_0$ if any $a_j = \emptyset_0$. We note $\mathcal{O}_{\mathbb{R}^0} = \mathbb{R}_{\mathbb{R}^0}$. We denote $a_{\mathbb{R}^n}$ by a_n, $a \in \mathbb{A}_0$, so $\emptyset_{\mathbb{R}^n} = \emptyset_n$.

Let X be any set. If $F \in \mathbb{A}_n,\ n > 0$, and $\varphi_1, \ldots, \varphi_n \in \mathbb{R}_X$, we can define

$F(\varphi_1, \ldots, \varphi_n)$ in $\mathbb{R}_X$ by

$$(F(\varphi_1, \ldots, \varphi_n))x = F(\varphi_1 x, \ldots, \varphi_n x), \ x \in X,$$

where the right-hand expression is computed in $\mathbb{A}_0$ using the rules given above. The domain of $F(\varphi_1, \ldots, \varphi_n)$ is

$$\{x \in X : x \in \text{dom } \varphi_1 \cap \ldots \cap \text{dom } \varphi_n, \ (\varphi_1 x, \ldots, \varphi_n x) \in \text{dom } F\}$$

because the two conditions together mean $F(\varphi_1 x, \ldots, \varphi_n x) \neq \emptyset_0$. The maps $(\mathbb{R}_X)^n \to \mathbb{R}_X : \varphi \mapsto F(\varphi)$ will be referred to as "substitution" maps. If $f : Y \to X$ is a map of sets, then

$$F(f^*\varphi_1, \ldots, f^*\varphi_n) = f^*(F(\varphi_1, \ldots, \varphi_n))$$

so substitution is compatible with transposition. Henceforth in calculations, such formulas will often be written just for the case $n = 1$, so the above formula would be $F(f^*\varphi) = f^*(F(\varphi))$.

Let X be any C^ω manifold. If $n > 0$, $F \in \mathbb{A}_n$ and $\varphi_1, \ldots, \varphi_n \in \mathcal{O}_X$, then $F(\varphi_1, \ldots, \varphi_n)$ is in $\mathcal{O}_X$ also. Thus substitution gives n-ary operations $(\mathcal{O}_X)^n \to \mathcal{O}_X$ on $\mathcal{O}_X$ $(n > 0)$, as well as on $\mathbb{R}_X$. We shall indicate how this way of looking at manifolds gives us an effective way to study the more general objects, resembling manifolds, that arise in control theory.

Let X be any set. If $\varphi, \psi \in \mathbb{R}_X$, we can define $\varphi + \psi = \alpha(\varphi, \psi)$, where $\alpha \in \mathbb{A}_2$ is the addition function $\alpha : \mathbb{R}^2 \to \mathbb{R}$, $\alpha(x, y) = x + y$. Then $\varphi + \psi \in \mathbb{R}_X$, dom $(\varphi + \psi) =$ dom $\varphi \cap$ dom ψ and $(\varphi + \psi)x = \varphi x + \psi x$ if $x \in$ dom $(\varphi + \psi)$. Of course $\varphi + \psi$ could have been defined the latter way instead, but here and throughout we want to stress the primary role played by substitution. We have a multiplication defined similarly by $\varphi\psi = \mu(\varphi, \psi)$ where $\mu(x, y) = xy, \ x, y \in \mathbb{R}$. The element $0_X\varphi = 0_1(\varphi), \ \varphi \in \mathbb{R}_X$, will be written 0φ. It is the function whose only scalar value is $0 \in \mathbb{R}$, and with the same domain as φ. Functions 0φ are called *nil* functions. They all have the form 0_Y, $Y \subset X$. We use the nil function 0_Y to encode the set Y in $\mathbb{R}_X$. If $0\varphi = 0_X$, we say φ is *global*. Note that $\varphi + 0\psi$ is the restriction of φ to dom $\varphi \cap$ dom ψ. If $\psi = \varphi + 0\psi$, ψ is a restriction of φ, and we write $\varphi \leq \psi$. (The reader might feel that we should write $\geq$ here.) This defines an order on $\mathbb{R}_X$. Taking $X = \mathbb{R}^0 = \{*\}$, we get an order on $\mathbb{A}_0$ with maximum element $\emptyset_0$ and such that distinct elements of $\mathbb{R} \subset \mathbb{A}_0$ are always incomparable.

If $\varphi, \psi \in \mathbb{R}_X$ and $\varphi + 0\psi = \psi + 0\varphi$, we say that φ and ψ *match*, because then φ and ψ give identical values at the points of dom $\varphi \cap$ dom ψ. If M is a subset of $\mathbb{R}_X$ and φ, ψ match whenever $\varphi, \psi \in M$, M is called *matching*. If H is any subset of $\mathbb{R}_X$, we let $\mathcal{M}(H)$ be $\{M \subset H, M \text{ matching}\}$. If $\varphi, \psi, \sigma \in \mathbb{R}_X$ and $\sigma \leq \varphi, \psi$, then φ and ψ match. Therefore any subset of $\mathbb{R}_X$ with a lower bound is an element of $\mathcal{M}(\mathbb{R}_X)$. Conversely, if M is matching, we can construct the greatest lower bound of M, written glb$M$ or $\underline{M}$, by *amalgamating* the elements of M, a process whose trivialities we shall now need to analyze closely.

Let $x \in X$. The inclusion $\{x\} \hookrightarrow X$ gives, by transposition, a map $\hat{x} : \mathbb{R}_X \rightarrow \mathbb{R}_{\{x\}}$. By identifying $\mathbb{R}_{\{x\}}$ with $\mathbb{A}_0$ as explained before, we get a map $\hat{x} : \mathbb{R}_X \rightarrow \mathbb{A}_0$. Since $\hat{x}$ is compatible with substitutions, we see that $\hat{x}(M) \in \mathcal{M}(\mathbb{A}_0)$ if $M \in \mathcal{M}(\mathbb{R}_X)$. However two distinct elements of $\mathbb{A}_0$ won't match unless one is $\emptyset_0$. Thus , if $N \in \mathcal{M}(\mathbb{A}_0)$, $\#N \leq 2$, and $\underline{N}$ is the lowest element of N unless $N = \emptyset$. Thus

$$\underline{N} = \begin{cases} \emptyset_0 & \text{if } N \subset \{\emptyset_0\} \\ \text{the unique element of } N \cap \mathbb{R} & \text{if } N \not\subset \{\emptyset_0\}. \end{cases}$$

We have

$$\hat{x}\underline{M} = \underline{M}(x) = \underline{M(x)} = \underline{\hat{x}(M)}$$

if $x \in X$.

We note that if $M, N \in \mathcal{M}(\mathbb{R}_X)$, then

$$\underline{M} + \underline{N} = \underline{M + N}$$

since, if $x \in X$,

$$(\underline{M} + \underline{N})(x) = \underline{M}x + \underline{N}x = \underline{Mx} + \underline{Nx} = \underline{Mx + Nx} = (\underline{M + N})(x).$$

The same is true for multiplication and also much more generally (cf (3.1.2) of [J1] for the probably ultimate version of such statements).

If $f : Y \rightarrow X$ and $M \in \mathcal{M}(\mathbb{R}_X)$, then

$$f^*(\underline{M}) = \underline{f^*(M)}$$

since, if $y \in Y$

$$(f^*(\underline{M}))y = \underline{M}(fy) = \underline{M(fy)} = \underline{(f^*M)y} = \underline{(f^*M)}y.$$

Thus transposition maps are compatible with glb.

A *function universe* on X, X a set, is a subset C of $\mathbb{R}_X$ such that the following holds :

1. If $n > 0$, $c \in C^n$ and $F \in \mathbb{A}_n$, then $F(c) \in C$,
2. If $M \in \mathcal{M}(C)$, $\underline{M} \in C$.
3. $0_X \in C$.

Thus a function universe contains 0 and is closed under substitution and amalgamation. We may call it an *analytic* function universe, because it is closed under analytic substitutions. Similarly "smooth" function universes may be defined. If C is a function universe on X we let $0_C = 0_X$, because 0_X is the neutral element for addition in C.

A function universe is a special case of the notion of topological universe as defined in [J1]. Topological universes are much more supple and useful than function universes. We can expect them to provide useful constructions for system theory also.

Let U be any set. Assume that whenever $n > 0$, $F \in \mathbb{A}_n$ and $u_1, \ldots, u_n \in U$, an element $F(u_1, \ldots, u_n)$ of U is given. (Even such a silly rule as $F(u_1, \ldots, u_n) = u_1$ would be a valid instance of this idea.) We then call U, with this special structure, a *weak universe*. We have already noticed that any function universe on a set X is a weak universe. We shall be able to save time by noticing that many notions about U expressed using points of X can be expressed entirely in terms of the weak universe structure on U. For instance, 0_X is given by $0_X(x) = 0,\ x \in X$, but also can be defined as the identity for addition on U. Likewise 1_X is the identity for multiplication on U. It is also clear that the statements "φ is nil, φ=glb$M$" can be explicated using only the weak universe structure on U. An *isomorphism* of weak universes $\varphi : U \to V$ is any bijection from U to V with the property

$$\varphi(F(u_1, \ldots, u_n)) = F(\varphi u_1, \ldots, \varphi u_n)$$

for $n > 0,\ F \in \mathbb{A}_n,\ u_1, \ldots, u_n \in U$. Notions like those above, which can be expressed with just the weak universe structure, will not be changed by isomorphisms. They will be referred to as *universe invariants*. We shall be considering isomorphic function universes on distinct sets, so universe invariance will be helpful.

A function *subuniverse* of a function universe C on X is any subset B of C which is also a function universe on X. We have the following obvious fact.

Lemma 3.1 *For any set X, $\mathbb{R}_X$ is a function universe. If X is a C^r manifold, $r \in \mathbb{N} \cup \{\infty, \omega\}$, then $\mathcal{O}_X$ is a function subuniverse of $\mathbb{R}_X$.* ♠

We mainly use Lemma 3.1 for $r = \omega$, but $r = \infty$ is also of some use.

Let X be any set and let H be a subset of $\mathbb{R}_X$. The intersection of any family of function subuniverses of $\mathbb{R}_X$ is a function subuniverse of $\mathbb{R}_X$, so there is a smallest function subuniverse $\mathcal{C}(H)$ of $\mathbb{R}_X$ that contains H. We note that if $X \neq \emptyset$, $\mathcal{C}(\emptyset_X)$ is the copy of $\mathbb{A}_0$ consisting of $\emptyset_X$ and the $a_X,\ a \in \mathbb{R}$. We shall call $\mathcal{C}(H)$ the *function universe* on X *generated by* H. An element of $\mathbb{R}_X$ is called H-*simple* if it can be written $F(h)$ with $F \in \mathbb{A}_n,\ n > 0,\ h \in (H \cup \{0_X\})^n$. Evidently, if φ is the greatest lower bound of a set of H-simple elements, then $\varphi \in \mathcal{C}(H)$. We have the following result.

Lemma 3.2 *Any element of $\mathcal{C}(H)$ is a greatest lower bound of H-simple elements.*

Proof: *Let U be the set of all greatest lower bounds of sets of H-simple elements. If $h \in H$, h is $H-$simple, since it is $z(h)$ with $z \in \mathbb{A}_1$ the identity map $z : \mathbb{R} \to \mathbb{R}$. Evidently $H \subset U \subset \mathcal{C}(H)$, so it will suffice to show that U is a function universe on X. Clearly $0_X \in U$ and U is closed under* glb. *If $u \in H \cup \{0_X\}$ and $F \in \mathbb{A}_1$, $v = F(u)$ is H-simple, and so is $G(v)$ for any*

$G \in \mathbb{A}_1$, *since* $G(v) = (G \circ F)(u)$. *At the expense of additional notational complexity, this same proof shows* $G(v_1, \ldots, v_n)$, *with* $G \in \mathbb{A}_n$, *is* H*-simple if* $v_1, \ldots, v_n$ *are* H*-simple. Since always* $F(\underline{M_1}, \ldots, \underline{M_n}) = \underline{F(M_1, \ldots, M_n)}$, U *is a function universe on* X. ♠

Let X be any subset of a C^ω manifold Y, $i : X \hookrightarrow Y$ the inclusion map. We call $\psi \in \mathbb{R}_X$ *analytic* if $\psi \in \mathcal{C}(i^*\mathcal{O}_Y)$. From Lemma 3.2, it follows easily that an element of $\mathbb{R}_X$ is analytic iff it is a glb of elements of $i^*\mathcal{O}_Y$. This means that locally on X, it is a restriction to X of some element of $\mathcal{O}_Y$.

Let U be a function universe on X. Define

$$\mathcal{T}_U = \{\text{dom } \varphi : \varphi \in U\}.$$

Then $n \mapsto \text{dom } n$ defines a one-to one map of $0U$ onto $\mathcal{T}_U$ since dom $\varphi=$ dom (0φ). We have therefore $\mathcal{T}_U = \{\text{dom } n : n \in 0U\}$. Moreover, if $\varphi, \psi \in U$ and $M \in \mathcal{M}(U)$, then

- $X = \text{dom } 0_X$,
- $\text{dom } (\varphi + \psi) = \text{dom } \varphi \cap \text{dom } \psi$,
- $\text{dom } \underline{M} = \bigcup\{\text{dom } \varphi : \varphi \in M\}$,

so we see that $\mathcal{T}_U$ is the collection of open sets for a topology on X that we call the U*-topology* on X. If $\varphi \in U$, then for any open $N \subset \mathbb{R}$, $\varphi^{-1}N = \text{dom } 0_N(\varphi)$. Thus $\varphi^{-1}N$ is a U-open subset of X and every $\varphi \in U$ is a continuous map $X \rightarrow \mathbb{A}_0$.

Let U be a function universe on the set X. Let

$$\Gamma_N U = \{\varphi \in \mathbb{R}_X : \text{dom}\varphi = N\},$$

where N is an open subset of X. Many readers will note that $N \rightarrow \Gamma_N U$ defines a sheaf of $\mathbb{R}$-algebras on X and wonder if sheaf theory might not be adequate for our concerns. Though some of our concerns can be addressed via sheaf theory, it has the drawback that whenever we consider any $\varphi \in U$, we must consider domφ along with φ. There is much more structure to U than just this sheaf, and this extra structure allows us to ignore domφ without losing any information about it. To see why it is very helpful to have a theory which keeps the domains out of sight, but not out of reach, one can contemplate the following example.

Example 3.3 *Let* $\epsilon > 0$ *be sufficiently small and let* $h : \mathbb{R}^2 \rightarrow \mathbb{A}_0$ *be defined for* $\sin^2 x + \sin^2 y > \epsilon$ *by*

$$h(x, y) = \sqrt{\sin^2 x + \sin^2 y - \epsilon}.$$

Then $h \in \mathbb{A}_2$. *Define* $\tilde{h}$ *on* $\mathbb{R}^5$ *by*

$$\tilde{h}(x, y, \alpha, \beta, t) = h(x + \alpha t, y + \beta t).$$

One could ask what $\mathrm{dom}\tilde{h}$ *in this example is like. From* $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ *we see* h *is doubly periodic of period* π*. Thus* h *can be regarded as a function on the torus* $T = \mathbb{R}^2/G$*, where* G *is the additive subgroup of* $\mathbb{R}^2$ *generated by* $(\pi, 0)$ *and* $(0, \pi)$*. Let* S *be the image in* T *of* $\{(x, y) \in \mathbb{R}^2 : \sin^2 x + \sin^2 y \leq \epsilon\}$*. (Thus* S *is a slightly deformed closed disc.) If* $\alpha, \beta \in \mathbb{R}$ *and* β/α *is irrational, every curve* $t \mapsto (x + \alpha t, y + \beta t)$*, when viewed as a curve in* T*, is a so-called "winding line" on* T*, i.e. has a dense image in* T *and so enters and leaves* S *infinitely many times. Many of the curves* $\gamma_{\alpha,\beta}$ *with* β/α *rational do not intersect* S *at all. In our context, functions like* $\tilde{h}$ *with such bizarre domains are very likely to arise. It is better to have a theory that does not require us to consider what points lie in the domain. Later on, we shall treat an example based on this* h *in a way that does not require us to worry about whether* β/α *is rational or irrational.*♠

The reader might ask how we contemplate doing calculations without considering domains and still have correct results. It is simply a matter of being slightly more careful with identities. Note, for example, that $\mathbb{A}_0$ satisfies the following identities:

$$\sin^2 u + \cos^2 u \equiv 1 + 0u$$

and

$$u/(v/w) \equiv (uw)/v + 0/w,$$

for $u, v, w \in \mathbb{A}_0$. (We note that $u/v = \div(u, v)$ where $\div \in \mathbb{A}_2$ is defined by $\div(x, y) = x/y,\ y \neq 0$.) Though this is obvious, it is good to point out the reasoning. The two sides of the equation considered equal each other (in $\mathbb{A}_0$) if $u, v, w \in \mathbb{R}$. If any letter u, v, w occuring has the value $\emptyset_0$, both sides equal $\emptyset_0$, because that letter occurs in both sides. These are also identities for every function universe on a set X. One proves that by evaluating both sides at an arbitrary $x \in X$. Any equality proven this way, as well as any which immediately results from such an equality, will be denoted with $\equiv$.

4 Universe spaces

A pair (X, C) is an (analytic) *universe space* if X is a set and C is an (analytic) function universe on X. Often we denote (X, C) by X and define $\mathcal{O}_X = C$. We always give X its $\mathcal{O}_X$-topology. Let X and Y be universe spaces. A map of sets $f : X \to Y$ is a *morphism* of universe spaces if $f^*\mathcal{O}_Y \subset \mathcal{O}_X$. Let us note the following properties of the map $h : U \to V$, where $U = \mathcal{O}_Y,\ V = \mathcal{O}_X$, defined by restricting f^* to $\mathcal{O}_Y$:

1. $h(F(\varphi)) = F(h\varphi_1, \ldots, h\varphi_n)$ whenever $n > 0$, $\varphi \in U^n,\ F \in \mathbb{A}_n$,
2. If $M \in \mathcal{M}(U), h(\underline{M}) = \underline{hM}$,
3. $h0_U = 0_V$.

If U and V are arbitrary function universes, a *morphism* of function universes $h : U \rightarrow V$ is any set map h of U into V which has the three properties listed above. Morphisms of function universes are not always induced by maps of the form f^*. For example, if U is a function subuniverse of V, then the inclusion map $h : U \hookrightarrow V$ is a morphism of function universes $U \rightarrow V$.

A morphism $h : U \rightarrow V$ is an *isomorphism* if it is bijective. One can show that h^{-1} is then also a morphism of function universes. Moreover if h is just an isomorphism of weak universes, it is an isomorphism of function universes.

Let U be a function universe on X. If $x, y \in X$ and $\varphi x = \varphi y$ whenever $\varphi \in U$, we say x and y are *U-kin*. If X is a universe space, $x, y \in X$ are *kin* if they are $\mathcal{O}_X$-kin.

Lemma 4.1 *Let X be a set, H a subset of $\mathbb{R}_X$. If $x, y \in X$, the following are equivalent :*

1. *x and y are $\mathcal{C}(H)$-kin,*
2. *$hx = hy$ for every h in H.*

Proof: *Since $\mathcal{C}(H) \supset H$, 1. $\Rightarrow$ 2.*
Assume 2., and let $U = \mathcal{C}(H)$ and

$$V = \{\varphi \in U : \varphi x = \varphi y\}.$$

Then $H \subset V \subset U$ and V is a function universe on X. Hence $V = U$, so x and y are U-kin.♠

Let X be any universe space. If x, y kin implies that $x = y$, the universe space X is *selective*. Let us relate this notion to the topology on X. A topological space W is called T_0 if, given any two points of W, there exists an open set which contains one of them and does not contain the other. We have the following :

Proposition 4.2 *A universe space is selective iff it is T_0.*
Proof: *Let $x, y \in X$, X a universe space, $x \neq y$. If X is selective, let $\varphi \in \mathcal{O}_X$, $\varphi x \neq \varphi y$. If say $\varphi y = \emptyset_0$, then $x \in \mathrm{dom}\ \varphi$, $y \notin \mathrm{dom}\ \varphi$. If $\varphi x, \varphi y \neq \emptyset_0$, this same observation applies if we use $1/(\varphi - (\varphi y)_X)$ instead of φ. This shows X is T_0.*
If conversely X is T_0, pick an open set U which contains for instance x but not y. Then $0_U x \neq 0_U y$. Thus x and y are not kin, and we see that X is selective.♠

A priori, we like universe spaces to be selective. However, control theory does freely produce universe spaces that aren't selective, and also universe spaces for which selectivity has to be demonstrated. Therefore we shall avoid making selectivity a general hypothesis, but we shall be interested in ways of constructing selective universe spaces from given universe spaces that are not selective, and to do so in such a way that the new function universe is isomorphic to the original one. We note that selectivity is not a universe invariant.

Let $\sim\subset X\times X$ be an equivalence relation. We shall say $\sim$ *implies* kinship if $x\sim y$ implies x and y are kin. Let $\sim$ imply kinship and let $\rho : X\to X/\sim$ be the equivalence class map. Then every φ in $\mathcal{O}_X$ is constant on every equivalence class, so we can write $\varphi = (\varphi/\sim)\rho$ for a unique $\varphi/\sim$ in $\mathbb{R}_{X/\sim}$, and $(\varphi/\sim)(x/\sim) = \varphi x$, where $x/\sim = \rho x$. Thus $\varphi\mapsto\varphi/\sim$ maps $\mathcal{O}_X$ one-to-one into $\mathbb{R}_{X/\sim}$. Its image is a function subuniverse $\mathcal{O}_{X/\sim}$ of $\mathbb{R}_{X/\sim}$. We can consider that X and $X/\sim$ have the same function universe $\mathcal{O}_X = \mathcal{O}_{X/\sim}$, though rigor will sometimes require that we keep them separate and use the basic formula $(\varphi/\sim)(x/\sim) = \varphi x$ that makes clear the logical distinction between φ and $\varphi/\sim$. In particular, since the kinship relation obviously implies kinship, we have the following.

Lemma 4.3 *Let X be a universe space, $\sim$ an equivalence relation on X that implies kinship, $\rho : X\to X/\sim$ the equivalence class map. Then ρ^* is an isomorphism $\rho^* : \mathcal{O}_{X/\sim}\to\mathcal{O}_X$. In particular, $\mathcal{O}_X\simeq\mathcal{O}_{X/\mathrm{kin}}$. Moreover, X/kin is selective.*♠

Let U be any function universe on a set X. A *point* of U is any morphism of function universes $U\to\mathbb{A}_0$. Then any $x\in X$ produces the point $\hat{x} : U\to\mathbb{A}_0$ of U defined by $\hat{x}(u) = u(x),\ u\in U$. We let SU denote the set of points of U. If $u\in U$, we can define $\hat{u}\in\mathbb{R}_{SU}$ by $\hat{u}(P) = P(u),\ P\in SU$. We have the convenient identity $\hat{u}(\hat{x}) = u(x),\ u\in U,\ x\in X$. If X is Hausdorff then the map $t_X : X\to S\mathcal{O}_X$, defined by $t_X x = \hat{x}$ is a homeomorphism (cf. Proposition 4.7).

Lemma 4.4 *Let U be a function universe. Then $u\mapsto\hat{u}$ defines an isomorphism t_U of U onto a function universe $\hat{U}$ on SU.*
Proof: *If $u,v\in U$ and $\hat{u} = \hat{v}$, then for $x\in X$, $u(x) = \hat{u}(\hat{x}) = v(x)$. If $n>0,\ F\in\mathbb{A}_n$ and $u\in U^n,\ F(t_U(u)) = t_U(F(u))$ because, if $P\in SU$,*

$$F(\hat{u})P = F(\hat{u}P) = F(Pu) = P(F(u)) = (\widehat{F(u)})(P).$$

(Here, and henceforth, such calculation is carried out only for $n=1$, since the case of arbitrary n is entirely similar.)
If $M\in\mathcal{M}(U),\ \mathrm{glb}(t_U M) = t_U(\mathrm{glb}M)$ since, if $P\in SU$,

$$(\mathrm{glb}\hat{M})P = \underline{\hat{M}(P)} = \underline{PM} = P\underline{M} = (\widehat{\underline{M}})(P).$$

Clearly $t_U 0_U = 0_{SC}$.♠

Proposition 4.5 *Let X be a universe space with $t_X : X\to S\mathcal{O}_X$ defined as before. Then $(t_X)^*$ is an isomorphism of the function universes $\mathcal{O}_{S\mathcal{O}_X}\to\mathcal{O}_X$.*
Proof: *Here we consider that $S\mathcal{O}_X$ is a universe space with the function universe $\widehat{\mathcal{O}_X}$ that consists of all $\hat{u}\in\mathcal{O}_X$. The proposition is immediate since every element of $\mathcal{O}_{S\mathcal{O}_X}$ can be written $\hat{u}$ for a (unique) $u\in U = \mathcal{O}_X$ and since $t_X^*\hat{u} = u$.*♠

Let X be a function universe. If $x \in X$, the *kinship class* of x is $\{y \in X : x$ and y are kin $\}$. Evidently, $\hat{x} = \hat{y}$ iff x and y are kin. Therefore, t_X induces an inclusion $X/\text{kin} \hookrightarrow S\mathcal{O}_X$, where X/kin denotes the set of kinship classes. The example below shows that, in general, $S\mathcal{O}_X$ is bigger than X/kin.

Example 4.6 *Let X be any infinite set. Let $\mathcal{O}_X$ consist of $\emptyset_X$ and the functions a_Y, where $a \in \mathbb{R}$ and $Y \subset X$ with $X \setminus Y$ finite. Then X is selective. Besides the elements $\hat{x}$ of $S\mathcal{O}_X$ there is one more. It is defined by*

$$P\emptyset_X = \emptyset_0, \; Pa_Y = a \text{ if } a \in \mathbb{R}, \; X \setminus Y \text{ finite.}\spadesuit$$

The condition X/kin Hausdorff will play an important role in what follows.

Proposition 4.7 *Let X be a universe space. Then the following are equivalent:*

1. *X/kin is Hausdorff.*
2. *$S\mathcal{O}_X$ is Hausdorff.*
3. *If $x, y \in X$ are not kin, there exists φ in $\mathcal{O}_X$ such that $\varphi x, \varphi y \in \mathbb{R}$, $\varphi x \neq \varphi y$.*

Moreover these conditions imply the canonical map $t_{X/\text{kin}} : X/\text{kin} \to S\mathcal{O}_X$ is a homeomorphism.

Before proving this, we note that 2. shows the conditions are all universe invariants of X and that 3. makes it easy to know when the conditions hold. We shall call X *quasi-Hausdorff* if the equivalent conditions of Proposition 4.7 hold. The space X in Example 4.6 is not quasi-Hausdorff. We note the following triviality, which shows us that the quasi-Hausdorff condition is easy to come by.

Proposition 4.8 *Let X be a universe space and assume that $\mathcal{O}_X$ is generated by global elements. Then X is quasi-Hausdorff.*
Proof: *Indeed, by Lemma 4.1, if $x, y \in X$ are not kin, there is a global φ of $\mathcal{O}_X$ with $\varphi x \neq \varphi y$.*$\spadesuit$

Proof of Proposition 4.7 :
3. $\Rightarrow$ 1. : Let $x', y' \in X/\text{kin}$, $x' \neq y'$ and let x, y be non-kin points of X such that $\rho x = x'$, $\rho y = y'$. Pick $\varphi \in \mathcal{O}_X$ with $\varphi x \neq \varphi y$ and both in $\mathbb{R}$. Let $\varphi = \rho^* \varphi'$, $\varphi' \in \mathcal{O}_{X/\text{kin}}$. Then also $\varphi' x' \neq \varphi' y'$. Let S and T be disjoint neighborhoods in $\mathbb{A}_0$ of $\varphi' x'$ and $\varphi' y'$ respectively. Then $(\varphi')^{-1} S$ and $(\varphi')^{-1} T$ are disjoint neighborhoods of x' and y' respectively.
1. $\Rightarrow$ 3. : Let A and B be disjoint open neighborhoods in X/kin of $x' = \rho x$ and $y' = \rho y$ respectively, and take $\varphi = \text{glb } \{0_A, 1_B\} \rho$.
2. $\Rightarrow$ 1. : $X/\text{kin} \subset S\mathcal{O}_X$ and has the relative topology.
1. $\Rightarrow$ 2. : To prove this and the rest of the proposition, we just need to show : 1. implies that $X/\text{kin} \to S\mathcal{O}_X$ is a homeomorphism. We can assume X is Hausdorff by replacing X by X/kin. We then know $X \to S\mathcal{O}_X$ is one-to-one. To show it is onto let $P \in S\mathcal{O}_X$ and let $Z_P = \{x \in X : \hat{x} \in \text{cl}\{P\}\}$.

Lemma 4.9 *If $\varphi \in \mathcal{O}_X$, then*

$$\text{dom } \varphi \cap Z_P \neq \emptyset \Leftrightarrow P\varphi \neq \emptyset_0.$$

Proof: *Let $R(\varphi) = \{P \in S\mathcal{O}_X : P\varphi \neq \emptyset_0\}$. (Then $R(\varphi)$ represents an arbitrary open set of $S\mathcal{O}_X$.) If $x \in \text{dom } \varphi \cap Z_P$, then $\hat{x} \in R(\varphi) \cap \text{cl}\{P\}$ so $P \in R(\varphi)$, i.e. $P\varphi \neq \emptyset_0$. Suppose* dom $\varphi \cap Z_P = \emptyset$. *If $x \in \text{dom } \varphi$, $\hat{x} \notin \text{cl}\{P\}$, so there is $n_x \in 0\mathcal{O}_X$ such that $\hat{x} \in R(n_x)$, $P \notin R(n_x)$, i.e. such that $n_x(x) = 0$, $Pn_x = \emptyset_0$. Then $\varphi = \text{glb}\{\varphi + n_x,\ x \in \text{dom } \varphi\}$, so $P\varphi = \text{glb}\{P(\varphi + n_x),\ x \in \text{dom } \varphi\} = \emptyset_0$.* ♠

Since $P0_X \neq \emptyset_0$, by Lemma 4.9 $Z_P \neq \emptyset$. If $x, y \in Z_P$, $x \neq y$, let A and B be disjoint open neighborhoods of x and y respectively. Then

$$\emptyset_0 = P(0_A + 0_B) = P0_A + P0_B = 0$$

a contradiction, so Z_P has just a single element x. (In general, when we do not assume 1., Z_P is a closed irreducible subset of X.) We see that if $\varphi \in \mathcal{O}_X$, $x \in \text{dom } \varphi \Leftrightarrow P\varphi \neq \emptyset_0$, so $\hat{x} \in \text{cl}\{P\}$. We have

$$\varphi x = \hat{\varphi}(\hat{x}) \in \text{cl}\{\hat{\varphi}(P)\} = \{P\varphi, \emptyset_0\}$$

so $\varphi x = P\varphi$ always, i.e. $P = \hat{x}$. Moreover, if $\varphi \in \mathcal{O}_X$, t_X identifies dom φ with $R(\varphi)$, so t_X is a homeomorphism. ♠

5 Vector fields and their trajectories

In order to study control systems on universe spaces, we need a concept of partially defined vector field.

Assume X is a universe space and let D be a map $\mathcal{O}_X \to \mathcal{O}_X$. We say that D satisfies the *chain rule* if, whenever $k > 0$, $F \in \mathbb{A}_k$ and $\varphi \in \mathcal{O}_X^k$, then

$$D(F(\varphi)) = \sum_{i=1}^{k} \partial_i F(\varphi) \cdot D\varphi_i,$$

where $\partial_i F$ is the i-th partial derivative of F. We also say that D *is a chain rule map of* $\mathcal{O}_X$. Obviously such a D is a universe invariant, i.e. if σ is an isomorphism of function universes, $\sigma^{-1} D \sigma$ satisfies the chain rule iff D does.

Lemma 5.1 *Let D be a chain rule map of $\mathcal{O}_X$. If $\varphi, \psi \in \mathcal{O}_X$, then*

1. $D\varphi = D\varphi + 0\varphi$

2. $D(\varphi + \psi) = D\varphi + D\psi$

3. $D(\varphi\psi) = \varphi D\psi + \psi D\varphi$

4. $D(0\varphi) = 0(D\varphi)$.

Proof: *1. Let* I *in* $\mathbb{A}_1$ *be the identity function on* $\mathbb{R}$*, i.e.* $I(x) = x$*. Then* $\partial_1 I = 1_1$*, the constant function on* $\mathbb{R}$ *of value 1. Hence*

$$D\varphi = D(I(\varphi)) = 1_1(\varphi)D\varphi \equiv D\varphi + 0\varphi.$$

Observe that 1. implies dom $D\varphi \subset$ dom φ, $\varphi \in \mathcal{O}_X$.
2. Note that $\partial_1\alpha = 1_2 = \partial_2\alpha$*, where* $\alpha : \mathbb{R}^2 \to \mathbb{R}$ *is the addition function and* $1_2 \in \mathbb{A}_2$ *the constant function on* $\mathbb{R}^2$ *of value 1. Hence*

$$D(\varphi + \psi) = D(\alpha(\varphi, \psi)) = 1_2(\varphi, \psi)D\varphi + 1_2(\varphi, \psi)D\psi = D\varphi + D\psi$$

because $1_2(\varphi, \psi)$ *has value 1 on* dom $D\varphi \cap$ dom $D\psi$.
3. If $z_1, z_2 : \mathbb{R}^2 \to \mathbb{R}$ *are the usual coordinates and* $\mu : \mathbb{R}^2 \to \mathbb{R}$ *the multiplication function, then* $\partial_1\mu = z_2$, $\partial_2\mu = z_1$*. Hence*

$$D(\varphi\psi) = D(\mu(\varphi, \psi)) = z_2(\varphi, \psi)D\varphi + z_1(\varphi, \psi)D\psi = \psi D\varphi + \varphi D\psi$$

since

$$z_2(\varphi, \psi) \equiv \psi + 0\varphi, \ z_1(\varphi, \psi) \equiv \varphi + 0\psi.$$

4. Finally

$$D(0\varphi) = D(0_1(\varphi)) = 0_1(\varphi)D\varphi = 0D\varphi.♠$$

From 4. of Lemma 5.1, $D0_X \in 0\mathcal{O}_X$, so $D0_X = 0_V$, where V is an open subset of X. If D is a chain rule map and $Dn = n + 0_V$ whenever $n \in 0\mathcal{O}_X$, we call D a (partially defined) *vector field* on X (or vector field of $\mathcal{O}_X$). We define dom $D = V$. If dom $D = X$, D is *globally defined* on X.

Example 5.2 *Let* $X = \mathbb{R}^n$ *and* $\mathcal{O}_X = \mathbb{A}_n$*. Let* V *be an open subset of* X*. Then the first order differential operator* $D = \sum_{i=1}^n \varphi_i \partial_i$*, where all* φ_i *are analytic functions with the domain* V*, is a partially defined vector field on* X *with domain* V*. It may be proved that every partially defined vector field on* $\mathbb{R}^n$ *has such a form.* ♠

Lemma 5.3 *Let* D *be a vector field on* X *and let* $D0_X = 0_V$*. Then for* $\varphi \in \mathcal{O}_X$, dom $D\varphi = V \cap$ dom φ.
Proof: *This follows immediately from*

$$0(D\varphi) = D(0\varphi) = 0\varphi + 0_V.♠$$

Note that vector fields are really universe invariants. We shall need the following technical result.

Lemma 5.4 *Let D be a vector field on X, $M \in \mathcal{M}(\mathcal{O}_X)$. Then $DM \in \mathcal{M}(\mathcal{O}_X)$ and $\underline{DM} = D\underline{M}$.*
Proof: *If $\varphi \leq \psi$, $D\varphi \leq D\psi$ since*

$$D\varphi + 0D\psi = D(\varphi + 0\psi) = D\psi.$$

Therefore $D\underline{M} \leq DM$, so $D\underline{M} \leq \underline{DM}$. Let $V = \mathrm{dom}\ D$. Then

$$0\underline{DM} = \underline{0DM} = \underline{D(0M)} = \underline{\{0\varphi + 0_V : \varphi \in M\}} = \underline{0M + 0_V}$$

$$= \underline{0M} + 0_V = 0\underline{M} + 0_V = D(0\underline{M}) = 0D\underline{M}$$

so $D\underline{M} = \underline{DM}$.♠

The following will help us define vector fields on subuniverses.

Lemma 5.5 *Let U be a function universe, $H \subset U$, D a vector field of U. Then the following are equivalent :*

1. *$D(H \cup \{0_U\}) \subset \mathcal{C}(H)$*
2. *$D(\mathcal{C}(H)) \subset \mathcal{C}(H)$.*

Proof: *Obviously 2. $\Rightarrow$ 1. Assume that 1. holds. Let $V = \{\varphi \in \mathcal{C}(H) : D\varphi \in \mathcal{C}(H)\}$. One shows easily that V is a function subuniverse of U. Evidently $H \subset V \subset \mathcal{C}(H)$ so $V = \mathcal{C}(H)$. Since $DV \subset \mathcal{C}(H)$, 2. holds.*♠

Example 5.6 *Let I be any interval in $\mathrm{I\!R}$ (open, closed or half-open), and let t be the usual coordinate on $\mathrm{I\!R}$. Observe that $(I, \mathcal{O}_I)$ is a universe space if $\mathcal{O}_I$ is the set of analytic elements of $\mathrm{I\!R}_I$ (cf. the definition after Lemma 3.2). Then d/dt, computed with one-sided derivatives at endpoints in I, is a vector field on I.* ♠

Let D be a vector field on the universe space X. A *trajectory* of D is a universe space morphism $\gamma : I \rightarrow X$ such that

$$\gamma^* D = (d/dt)\gamma^*.$$

It follows that then

$$\mathrm{im}\gamma \subset \mathrm{dom} D$$

because

$$(D0_X)\gamma = \gamma^* D0_X = d/dt\ \gamma^* 0_X = d/dt\ 0_I = 0_I$$

and $\mathrm{dom} D0_X$ is $\mathrm{dom} D$.

When our universe space is a manifold, our definition of trajectory is equivalent to the standard one. When $c \in I$ and is the initial (resp. terminal) endpoint of I, we say the trajectory γ *emanates from* (resp. *terminates at*) x if $\gamma c = x$. A *local trajectory* through $x \in X$ is a trajectory $\gamma : (-\epsilon, \epsilon) \rightarrow X$ defined for some $\epsilon > 0$ and with $\gamma 0 = x$. We consider any two such to be equivalent if they agree on some neighborhood of 0, and we then say the two local trajectories are the same.

Existence and uniqueness of local trajectories is not guaranteed in general, but in many cases these properties hold.

Example 5.7 *Local trajectories exist uniquely for any vector field on an analytic manifold.*♠

Example 5.8 *Let* $X = [0, \infty)$ *with* $\mathcal{O}_X$ *the analytic elements of* $\mathbb{R}_X$*. Then the identity map of* X *is a trajectory of* d/dt *emanating from* 0*, but there is no local trajectory of* d/dt *through* 0.♠

Example 5.9 *Let* S *be a subset of* $\mathbb{R}$ *and let* $\mathbb{R}_{2S}$ *consists of all pairs* $(a, +)$ *and* $(a, -)$*,* $a \in \mathbb{R}$*, modulo the equivalence relation generated by* $(a, +) \sim (a, -)$ *if* $a \notin S$*. Then* $\mathbb{R}_{2S}$ *is* $\mathbb{R}$ *with* S *occuring in a double layer. We have two natural maps* $u_+ = u_{+,S},\ u_- = u_{-,S}$ *of* $\mathbb{R}$ *into* $\mathbb{R}_{2S}$ *given by*

$$u_+(a) = [(a, +)],\ u_-(a) = [(a, -)].$$

Let

$$\mathcal{O}_{\mathbb{R}_{2S}} = \{\varphi \in \mathbb{R}_{\mathbb{R}_{2S}} : u_+^* \varphi, u_-^* \varphi \in \mathbb{A}_1\}.$$

There is a unique derivation D *of* $\mathcal{O}_{\mathbb{R}_{2S}}$ *such that* $u_\epsilon^* D = d/dt\ u_\epsilon^*$*,* $\epsilon = \pm$*, and it is a globally defined vector field on* $\mathbb{R}_{2S}$*. If we let* $S = (0, \infty)$*, then* u_+ *and* u_- *are trajectories of* D *through* $0 = [(0, +)] = [(0, -)]$ *that do not agree in any neighborhood of* $0 \in \mathbb{R}$.♠

Let U be a function universe on X. Let $\Gamma U = \{u \in U : 0u = 0_U\}$; ΓU consists of global functions on X. Call U *global* if ΓU generates U. The property of being global is a universe invariant. If $n \in 0U$, let $\Gamma_n U = \{u \in U : 0u = n\}$, so that $\Gamma U = \Gamma_{0_U} U$. Let also $U_{|n} = \{u \in U : \text{dom } u \subset \text{dom } n\}$. Note that $U_{|n}$ is a function universe on dom n, but it is not a function subuniverse of U. Call n *full* if the function universe $U_{|n}$ is global.

Let X be a universe space and let $x \in X$. Call x *full* if there exists an open neighborhood N of x such that $0_N \in \mathcal{O}_X$ is full. Thus $x \in X$ is full if every element of $\mathcal{O}_X$ whose domain is contained in a certain neighborhood of x can be written locally using functions everywhere defined on that same neighborhood. We note that $0 \in \mathbb{R}_{2(0,\infty)}$ (cf. Example 5.9) is not full. Call X *locally full* if every point of X is full.

The following trivial remark shows that X being locally full is universe invariant. This will be useful when we consider several universe spaces X all having the same function universe $\mathcal{O}_X$.

Lemma 5.10 *Let X be a universe space. The following are equivalent :*

1. *X is locally full,*

2. *$0_X = \mathrm{glb}\{n \in 0\mathcal{O}_X : n$ is full$\}$.*♠

Proposition 5.11 *Assume that the function universe X is selective. Let D be a vector field on X and let $x \in \mathrm{dom}D$. Assume that x is full and suppose σ and τ are local trajectories of D through x. Then there is $\epsilon > 0$ such that $\sigma = \tau$ on $(-\epsilon, \epsilon)$.*

Proof: *Let σ and τ be trajectories of D with $\sigma 0 = x = \tau 0$. Choose an open neighborhood N of x with $\mathcal{O}_{X|0_N} = \mathcal{C}(\Gamma_N \mathcal{O}_X)$ where $\Gamma_N \mathcal{O}_X = \Gamma_{0_N} \mathcal{O}_X$ is understood as a set of functions on N. Let I be an open interval containing 0 such that $\sigma(I),\ \tau(I) \subset N$. Since X is selective, it will suffice to show that whenever $\varphi \in \mathcal{O}_X,\ \varphi\sigma t = \varphi\tau t$ whenever $t \in I$. We have*

$$\varphi(\sigma t) = (\varphi + 0_N)(\sigma t)$$

and the same for τ if $t \in I$, so we can assume $\varphi \in \mathcal{O}_{X|0_N}$. By Lemma 4.1 we can assume $\varphi \in \Gamma_N(\mathcal{O}_X)$. Since σ is a trajectory of D,

$$(D^n \varphi)\sigma = (d/dt)^n (\varphi\sigma)$$

for any $n \geq 0$, and the same for τ. Evaluating at $t = 0$, we find $\varphi\sigma$ and $\varphi\tau$ have the same power series at 0, and are defined on all of I. By analyticity, $\varphi\sigma = \varphi\tau$ on I.♠

It will be important in our applications to know when a trajectory γ defined in the interval $[a, b]$, $a < b$, is uniquely determined by γa. The following example shows it is not enough to assume that X is locally full.

Example 5.12 *Let $X = \mathbb{R}_{2[0,\infty)}$. Then u_+ and u_- give distinct trajectories of D on $[-1, 0]$ emanating from the point $[(-1, +)] = [(-1, -)]$.* ♠

In this example X is selective, but not Hausdorff.

Theorem 5.13 *Let X be a locally full Hausdorff universe space. Let $x \in$ dom D, D a vector field on X, and let $a, b \in \mathbb{R}$, $a < b$. Then there is at most one trajectory $\gamma : [a, b] \to X$ with $\gamma(a) = x$.*

Proof: *Let $\gamma, \delta : [a, b] \to X$ be trajectories of D with $\gamma a = x = \delta a$. Let $S = \{t : a \leq t \leq b,\ \gamma t \neq \delta t\}$ and let $c = \inf\ S$. It will suffice to show $c = b$ and c is not in S. If $c \in S$, then $a < c$ and there are neighborhoods U of γc and V of δc with $U \cap V = \emptyset$. If $a \leq t < c$ and t is near enough to c, then $\gamma t = \delta t$ is in U and V, a contradiction. Therefore $c \notin S$. From Proposition 5.11, applied to $x = \gamma c = \delta c$, unless $c = b$ we get, for some $\epsilon > 0$, that $\gamma(t) = \delta(t)$ if $t \in [c, c + \epsilon)$, so these t are not in S. Therefore $c = b$.* ♠

If D is any vector field on X and $x \in \mathrm{dom}D$, a *forward local trajectory at x* is given by any trajectory $\gamma : [0, \epsilon) \rightarrow X$ of D with $\gamma 0 = x$. We consider any two such to define the same forward local trajectory at x if they coincide on $[0, \eta)$ for some (sufficiently small) $\eta > 0$. We say *D has forward local trajectories* if D has a forward local trajectory at x for every $x \in \mathrm{dom}D$. We say *forward local trajectories* of D *are unique* if given any $x \in \mathrm{dom}D$ and trajectories $\gamma, \delta : [0, \epsilon) \rightarrow X$ with $\gamma 0 = x = \delta 0$, γ and δ define the same forward local trajectory of D at x. We have the following result proved just as Proposition 5.11 was proved.

Lemma 5.14 *If the assumptions of Proposition 5.11 hold, there is at most one forward local trajectory of D at x.* ♠

If $a \in \mathbb{R}$, any map γ of the degenerate interval $[a, a]$ (given by γa) is a trajectory of any vector field D provided that $\gamma a \in \mathrm{dom}D$. The following lemma is used implicitly at points of the sequel.

Lemma 5.15 *Let X be a universe space and D a partially defined vector field on X. Let*

$$a = x_0 \leq x_1 \leq \ldots \leq x_m = b$$

be elements of $\mathbb{R}$, where $m \geq 0$, and let γ be a map of $[a, b]$ into X. The following are equivalent:
1) γ is a trajectory of D.
2) For each $i = 1, \ldots, m$, $\gamma_{|[x_{i-1}, x_i]}$ is a trajectory of D.
Proof: *Since γ a trajectory of D implies $\mathrm{im}\gamma \subset \mathrm{dom}D$, it is apparent that the possibility $x_{i-1} = x_i$ for any i presents no difficulties, so we shall assume $x_{i-1} < x_i$, $1 \leq i \leq m$. For each $i = 1, \ldots, m$, let $\iota_i : [x_{i-1}, x_i] \rightarrow [a, b]$ be the inclusion. Then, for any i*

$$(d/dt)(\gamma\iota_i)^* = \iota_i^*(d/dt)\gamma^*$$

and

$$(\gamma\iota_i)^* D = \iota_i^*(\gamma^* D).$$

Assume 1). Then for every i

$$(d/dt)(\gamma\iota_i)^* = (\gamma\iota_i)^* D,$$

i.e. 2) is true. Assume 2). Then for every i

$$(d/dt)\gamma^*\varphi_{|[x_{i-1}, x_i]} = \gamma^* D\varphi_{|[x_{i-1}, x_i]}$$

whenever $\varphi \in \mathcal{O}_X$, i.e. 1) is true. ♠

Lemma 5.16 *Assume $\mathcal{O}_X$ is generated by G and that D is a vector field on X. Then $\gamma : I \to X$ is a trajectory of D iff the following both hold:*
1) $\text{im}\gamma \subset \text{dom}D$,
2) $\gamma^* Dg = d/dt\ \gamma^* g$ *for every* $g \in G$.
Proof: *It is enough to prove the "if" part. Define $U \subset \mathcal{O}_X$ by*

$$U = \{\varphi \in \mathcal{O}_X : \gamma^* D\varphi = d/dt\ \gamma^*\varphi\}.$$

Then $0_X \in U$ by 1), and $G \subset U$ also. If $n > 0$, $F \in \mathbb{A}_n$, and $\varphi_1, \ldots, \varphi_k \in U$, the chain rule implies that

$$\gamma^* DF(\varphi) = d/dt\ \gamma^*(F(\varphi)).$$

Also if $M \in \mathcal{M}(U)$, then $\underline{M} \in \mathcal{M}(U)$ by Lemma 5.4. Thus U is a function universe with $G \subset U$, so $U = \mathcal{O}_X$, whence γ is a trajectory of D. ♠

6 Systems on universe spaces

Let X be a universe space and let Ω a set. A *field* on X *with commands* Ω, or Ω-*field*, is a family $\mathcal{D} = (D_\omega)_{\omega\in\Omega}$ of (partially defined) vector fields on X. We shall say that the Ω-field $\mathcal{D}$ *has forward trajectories* if D_ω has forward trajectories for every $\omega \in \Omega$. We shall say that *forward local trajectories* of $\mathcal{D}$ *are unique* if for every $\omega \in \Omega$ forward local trajectories of D_ω are unique. If $\mathcal{D}$ has forward local trajectories and they are unique, we call $\mathcal{D}$ a *dynamics* on X.

An *observation structure* on X is any subset H of $\mathcal{O}_X$ such that

$$\bigcup_{\varphi\in H} \text{dom}\ \varphi = X.$$

Elements of H are called *observation functions*. Usually H will consist of a finite number of observation functions $h_1, \ldots, h_r$. It is important not to assume that the elements of H all have the same domain, for we view them as representing various measurement devices whose ranges might be distinct.

Let us fix a set of commands Ω. A *quasi-system* is a 3-tuple $\Sigma = (X, \mathcal{D}, H)$ where X is a universe space called *state space*, $\mathcal{D} = (D_\omega)_{\omega\in\Omega}$ is an Ω-field on X and H is an observation structure on X. We write $\mathcal{O}_\Sigma$ for $\mathcal{O}_X$, X_Σ for X, and say that Σ is an Ω-*quasi-system.* If $\mathcal{D}$ is a dynamics, we shall say Σ is *dynamic*.

We call a quasi-system *bidynamic* if given any control ω and any $x \in \text{dom}D_\omega$, there is a unique local trajectory of D_ω through x. It is equivalent to say that the system Σ and its negative $-\Sigma$, with vector fields $-D_\omega$, are both dynamic. Most of our results do not require that a quasi-system be bidynamic, so we get clearer proofs and greater generality by not making this stronger assumption.

We note that X is naturally a topological space. We assume our quasi-systems are *analytic*, i.e. that X is an analytic universe space, unless stated otherwise. We shall say that Σ *has trajectories* if $\mathcal{D}$ has local forward trajectories. Similarly, we shall say that *trajectories* of Σ *are locally unique* if local forward trajectories of $\mathcal{D}$ are unique. We shall say Σ is *selective* if X is selective. If Σ is selective and dynamic, we call Σ a *system.* The concept of system abstracts the essential elements of the following very standard idea (compare with Section 2).

Example 6.1 *Let $X = M$ be an analytic manifold, $\mathcal{O}_M$ the universe of analytic functions on M. Consider the following set of equations:*

$$\dot{x}(t) = f(x(t), u(t)),$$

$$y(t) = h(x(t)),$$

where $t \in \mathbb{R}$, $x(t) \in M$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^r$, $f(\cdot, \omega)$ is a (globally defined) vector field on M for $\omega \in \mathbb{R}^m$, and $h = (h_1, \ldots, h_r)$, each h_j a (globally defined) analytic function on M. Let $D_\omega = f(\cdot, \omega)$ and $H = (h_i)_{i=1,\ldots,r}$. Put $\Omega = \mathbb{R}^m$. Then $X, \mathcal{D}, H$ form a system.♠

The *control* function u that appears in the above example allows us to change vector fields. Since, in general, we do not assume any structure in Ω, controls will be piecewise constant functions of time with values in Ω. Controls defined on finite intervals $[0, T]$ will be called *experiments of duration* T. An experiment e will be denoted by the sequence $((t_1, \omega_1), \ldots, (t_k, \omega_k))$. It means that we apply D_{ω_1} for time t_1, D_{ω_2} for time t_2 and so on. Let $t_0 = 0$ and $T_i = \sum_{s=0}^{i} t_s$. Then $T_k = T_e$, the duration of e. We call ω_i the *i-th command* of e. Denote by E the set of all experiments. We assume that all $t_i \geq 0$, so $T_e \geq 0$. If $e \in E$ then a *trajectory* of e is any continuous map $\gamma : [0, T_e] \to X$ such that $\gamma_{|[T_{i-1}, T_i]}$ is a trajectory of D_{ω_i}, $1 \leq i \leq k$. Since we allow $T_{i-1} = T_i$, any $\sigma : [t, t] \to X$ will be called a trajectory of any vector field D, provided $\sigma t \in \text{dom } D$. If γ is a trajectory of e with command ω on (a, b), $0 \leq a < b \leq T_e$, then D_ω is defined at γa and γb. Moreover ω is the last command at γa and the first at γb. The trajectory $\gamma : [0, T_e] \to X$ is said to *emanate* from $x \in X$ if $x = \gamma 0$, and it *terminates* at γT_e. We call the experiment e *apt* for $x \in X$ if whenever $0 \leq t \leq T_e$, $e_{\leq t}$, which is e restricted to $[0, t]$, has a unique trajectory emanating from x. We also then say that x is *apt* for e. When this is the case, we define xe, the *outcome* of x under e, by $xe = \gamma T_e$, where γ is the (unique) trajectory of e emanating from x. If $x, y \in X$, and x is apt for e iff y is for each each $e \in E$, we say that x and y are *isomobile.*

Let us notice that if X is locally full and Hausdorff, an experiment e is apt for x iff a trajectory of e emanating from x exists (see Theorem 5.13). We shall need this later; in this section non-uniqueness of trajectories can cause e not to be apt for x.

If e is any experiment and we wish to show that a given morphism $\gamma : I \to X$ is a trajectory of e, we shall usually avoid trivial complications by assuming that

e has just a single command ω. It is expected that when we exercise this option, the reader will encounter no difficulty in extending the argument to arbitrary experiments.

Let $\Sigma = (X, \mathcal{D}, H)$ and $\Sigma' = (X', \mathcal{D}', H')$ be quasi-systems with the same set of commands Ω. An *Ω-morphism* $\mu : \Sigma \to \Sigma'$ is a map $\mu : X \to X'$ that satisfies the following conditions :

1. The map μ is a morphism of universe spaces.

2. $D_\omega \mu^* = \mu^* D'_\omega$ whenever $\omega \in \Omega$.

We make no assumptions about the effect of μ^* on the observation structures.

Lemma 6.2 *Let Σ, Σ' be quasi-systems with a common set of commands Ω, and let $\mu : \Sigma \to \Sigma'$ be an Ω-morphism. If $e \in E$ and γ is a trajectory of e for Σ, then $\gamma' = \mu\gamma$ is a trajectory of e for Σ'.*
Proof: *As suggested above, we argue only the case when e has a single command ω. Let $t \in [0, T_e]$. Then*

$$(d/dt)(\gamma')^* = d/dt\ \gamma^* \mu^* = \gamma^* D_\omega \mu^* = \gamma^* \mu^* D'_\omega = (\gamma')^* D'_\omega. \spadesuit$$

Let $e \in E$. If $\varphi \in \mathrm{I\!R}_X$, define $e^*\varphi$ in $\mathrm{I\!R}_X$ by

$$(e^*\varphi)x = \begin{cases} \varphi(xe) & \text{if } e \text{ is apt for } x, \\ \emptyset_0 & \text{otherwise.} \end{cases}$$

If γ is a trajectory of e terminating at x and δ a trajectory of e' emanating from x, then $\gamma\delta$, which is γ followed by δ, is a trajectory of ee'. Thus if $e_1, e_2 \in E$, then $(e_1e_2)^*\varphi = e_1^*(e_2^*\varphi)$ where e_1e_2 is e_1 followed by e_2. The function φ serves as a model of a measuring device, and $(e^*\varphi)x$ is just the measured value at the end of the trajectory. Once the trajectory leaves the range of the device, we assign value $\emptyset_0$.

We leave the following result to the reader.

Lemma 6.3 *Let $e \in E$.*

1. *If $n > 0$, $F \in \mathrm{I\!A}_n$, $\varphi_1, \ldots, \varphi_n \in \mathrm{I\!R}_X$, then*

$$e^*(F(\varphi_1, \ldots, \varphi_n)) = F(e^*\varphi_1, \ldots, e^*\varphi_n).$$

2. *If $M \in \mathcal{M}(\mathrm{I\!R}_X)$, $e^*\underline{M} = \underline{e^*M}$.* $\spadesuit$

Thus $e^* : \mathrm{I\!R}_X \to \mathrm{I\!R}_X$ is almost a morphism. The condition $e^*0_X = 0_X$ is in general not satisfied.

Let $\Sigma = (X, \mathcal{D}, H)$ be any quasi-system, $\varphi \in \mathrm{I\!R}_X$, $x, y \in X$. We shall say that φ *can't experimentally distinguish* x from y if $\varphi x = \varphi y$ and $(e^*\varphi)x = (e^*\varphi)y$ for all $e \in E$. (It does not then follow that x and y are isomobile, since, for example, $\emptyset_X$ cannot distinguish any point of X from any other.) Points x and y in X are called Σ-*indistinguishable* if no element of H can experimentally distinguish them. If x and y are Σ-indistinguishable, we cannot tell x from y by performing experiments and observing functions $h \in H$.

Proposition 6.4 *Let Σ be a quasi-system. If $x, y \in X$ are Σ-indistinguishable, they are isomobile.*

Proof: *Suppose x and y are not isomobile. Let e be an experiment such that x is apt for e and y is not. Let φ be a function in H such that $xe \in \mathrm{dom}\ \varphi$. Then $(e^*\varphi)y = \emptyset_0$, so φ can experimentally distinguish x and y. Thus x and y are not Σ-indistinguishable.* ♠

Our definition eliminates the role that completeness or analyticity play in the definition of classical indistinguishability. It is easy to see that Σ-indistinguishability is always an equivalence relation. This is also the case for C^∞ systems on smooth manifolds or on "smooth" universe spaces, which may be defined like analytic ones using C^∞ functions for substitution operators.

We say that Σ is *observable* if no two distinct points of the state space X are Σ-indistinguishable. A standard problem in system theory is to transform an unobservable system into an observable one while preserving its input/output behavior. We shall address this problem in the next sections.

The *observation universe* of Σ, denoted by $\mathcal{U}(\Sigma)$, is the smallest function subuniverse of $\mathcal{O}_X$ containing H and closed under the action of the vector fields in $\mathcal{D}$. It is a natural generalization of the concepts of observation space and observation algebra (cf. [B2],[S1]). We shall say that x and y in X are *infinitesimally indistinguishable* if $\varphi x = \varphi y$ for all φ in $\mathcal{U}(\Sigma)$. This means that x and y are kin for the universe space $(X, \mathcal{U}(\Sigma))$.

Lemma 6.5 *Let Σ be any quasi-system. Then $\mathcal{U}(\Sigma)$ is generated by*

$$\{D_{\omega_k} \dots D_{\omega_1} h : k \geq 0,\ \omega_1, \dots, \omega_k \in \Omega,\ h \in H\}.$$

Proof: *Let S denote the set of these elements of $\mathcal{O}_\Sigma$. Evidently $S \subset \mathcal{U}(\Sigma)$, so $\mathcal{C}(S) \subset \mathcal{U}(\Sigma)$. Let $\omega \in \Omega$. Then $D_\omega S \subset S$. Also $D_\omega 0_X \in \mathcal{C}(S)$ since it is*

$$D_\omega \underline{0H} = \underline{D_\omega(0H)} = \underline{0 D_\omega H}$$

and $D_\omega H \subset S$. Therefore

$$D_\omega(\mathcal{C}(S)) \subset \mathcal{C}(S)$$

by Lemma 5.5. Thus $H \subset \mathcal{C}(S)$ and $\mathcal{C}(S) \subset \mathcal{U}(\Sigma)$ is closed under the action of vector fields in $\mathcal{D}$, so $\mathcal{C}(S) = \mathcal{U}(\Sigma)$. ♠

Theorem 6.6 *Assume Σ is a dynamic quasi-system. If $x, y \in X$ are Σ-indistinguishable, then they are infinitesimally indistinguishable.*

Lemma 6.7 *Assume Σ is a dynamic quasi-system. Let $x, y \in X$, $\varphi \in \mathcal{O}_X$, $\omega \in \Omega$. If φ cannot experimentally distinguish x from y, then also $D_\omega \varphi$ can't experimentally distinguish x from y.*
Proof: *Let $e \in E$. To show $(e^* D_\omega \varphi)x = (e^* D_\omega \varphi)y$, let T be the duration of e and, for $t \geq T$, let e_t be the experiment e followed by $(t - T, \omega)$. Then $(e_t^* \varphi)x = (e_t^* \varphi)y$ since φ can't experimentally distinguish x from y. Assume $(e^* D_\omega \varphi)x \neq \emptyset_0$. Then e is apt for x, $xe \in \mathrm{dom} D_\omega \varphi$ and $(e^* D_\omega \varphi)x = (D_\omega \varphi)(xe)$. Also φ is defined at xe, so $(e^* \varphi)x \neq \emptyset_0$. Then $(e^* \varphi)y \neq \emptyset_0$, so e is apt for y. Let γ be the trajectory of e_t emanating from x. It will exist uniquely if $t \geq T$ is sufficiently close to T. Then*

$$(e^* D_\omega \varphi)x = (D_\omega \varphi)xe = d/dt_{|t=T+}(\varphi \circ \gamma)t = d/dt_{|t=T+}(e_t^* \varphi)x = (e^* D_\omega \varphi)y$$

because the last expression can be computed with the same calculation, using the unique trajectory of e_t emanating from y. If $(e^* D_\omega \varphi)x = \emptyset_0$, then $(e^* D_\omega \varphi)y = \emptyset_0$, as we see by interchanging x and y in what was proved already. To show $(D_\omega \varphi)x = (D_\omega \varphi)y$, observe that $(e^* D_\omega \varphi)x = (e^* D_\omega \varphi)y$ where e is the experiment $(0, \omega)$, and that $(e^* D_\omega \varphi)x = (D_\omega \varphi)x$. ♠

Proof of Theorem 6.6
Let $x, y \in X$ be Σ-indistinguishable. By Lemmas 6.5 and 4.1 it will suffice to show $\varphi x = \varphi y$ for each φ in the set of Lemma 6.5. By assumption, nothing in H can experimentally distinguish x from y. From Lemma 6.7 , no φ under consideration can experimentally distinguish x from y, so $\varphi x = \varphi y$.♠

We call Σ *infinitesimally observable* if no two points of X are infinitesimally indistinguishable.

Corollary 6.8 *If Σ is dynamic and infinitesimally observable, then it is also observable.* ♠

The reverse implication is false, because our vector fields and output functions are partially defined (cf. Example 7.7).

Theorem 6.9 *Assume that a quasi-system Σ is dynamic. Let $\Sigma_{\mathcal{U}} = (X, \mathcal{D}, \mathcal{U}(\Sigma))$. Then $x, y \in X$ are Σ-indistinguishable iff they are $\Sigma_{\mathcal{U}}$-indistinguishable.*
Proof: *Since $H \subset \mathcal{U}(\Sigma)$, $\Sigma_{\mathcal{U}}$-indistinguishable implies Σ-indistinguishable. Conversely, assume x and y are Σ-indistinguishable. Let*

$$U = \{\varphi \in \mathcal{U}(\Sigma) : \varphi \text{ can't experimentally distinguish } x \text{ from } y\ \}.$$

Then, since x and y are isomobile, U is a function subuniverse of $\mathcal{U}(\Sigma)$ by Lemma 6.3. By Lemma 6.7, $D_\omega U \subset U$ for any $\omega \in \Omega$. Therefore every function in the set of Lemma 6.5 is in U. Since $\mathcal{U}(\Sigma)$ is generated by such functions, $U = \mathcal{U}(\Sigma)$. Thus elements of $\mathcal{U}(\Sigma)$ can't experimentally distinguish x from y, whence x and y are $\Sigma_{\mathcal{U}}$-indistinguishable.♠

Corollary 6.10 *A dynamic quasi-system* Σ *is observable iff the system* $\Sigma_{\mathcal{U}}$ *is observable.*♠

Each point $x \in X$ defines the *response map* F_x of the system Σ :

$$F_x : E \to \mathbb{R}_H, \quad (F_x e)h = (e^* h)x.$$

If x, y are Σ-indistinguishable, then $F_x = F_y$, so the set of response maps $\Phi = \{F_x : x \in X\}$ can be identified with the set of equivalence classes modulo Σ-indistinguishability. When transforming the system, we usually want to preserve the set Φ of response maps. So far we can say the following.

Lemma 6.11 *Let* $\mu : \Sigma \to \Sigma'$ *be an* Ω*-morphism of quasi-systems,* x *a state of* Σ*,* h' *an observation function of* Σ'*. Then, if* $e \in E$ *and* x *and* μx *are both apt for* e*, or if neither is apt for* e*,*

$$(F_x e)(h'\mu) = (F_{\mu x} e)h'.$$

Proof: *Let* T *be the duration of* e*. Using Lemma 6.2 and its notation, we get :*

$$(F_x e)(h'\mu) = (e^*(h'\mu))x = h'(\mu(xe)) = h'(\mu(\gamma T)) = h'(\gamma' T)$$

$$= h'((\mu x)e) = (F_{\mu x} e)h'.♠$$

7 Deriving systems from quasi-systems; quotient structures

Let Σ be any quasi-system. States x, y of Σ are called Σ-*kin* if $\varphi x = \varphi y$ whenever $\varphi \in \mathcal{O}_\Sigma$. This defines an equivalence relation called Σ-*kinship* (or *kinship*). The map $\rho : X \to X/\Sigma\text{-kin}$ is called the Σ-*kinship map*, and we set $\rho x = x/\Sigma\text{-kin}$ and $X' = X/\Sigma\text{-kin}$. From Lemma 4.3 we get a function universe $\mathcal{O}_{X'}$ such that $\rho^* : \mathbb{R}_{X'} \to \mathbb{R}_X$ restricts to an isomorphism $\psi : \mathcal{O}_{X'} \to \mathcal{O}_X$. Because ψ is an isomorphism, $\mathcal{D}' = (\psi^{-1} D_\omega \psi)_{\omega \in \Omega}$ is an Ω-field for the universe space X'.

We let $\Sigma/\text{kin} = (X', \mathcal{D}', \psi^{-1} H)$. This is a selective quasi-system, and $\rho : X \to X'$ is an Ω-morphism of quasi-systems $\Sigma \to \Sigma/\text{kin}$.

We can reverse Theorem 6.6 by adding some assumptions. Note that the hypotheses in the following result hold if the elements of H and the $D_\omega, \omega \in \Omega$ are all globally defined on X (cf. Proposition 4.8).

Theorem 7.1 *Assume that the universe space* $(X, \mathcal{U}(\Sigma))$ *is locally full and quasi-Hausdorff. If* $x, y \in X$ *are isomobile and infinitesimally indistinguishable, they are* Σ*-indistinguishable.*

Proof: *Let* X_e *denote the universe space* $(X, \mathcal{U}(\Sigma))$*. By restriction,* $\mathcal{D}$ *defines an* Ω*-field* $\mathcal{D}_e$ *on* X_e*. Let* Σ_e *be the quasi-system* $(X_e, \mathcal{D}_e, H)$*. By hypothesis,*

Σ_e/kin *is Hausdorff and locally full, so trajectories of experiments are unique for* Σ_e/kin *by Theorem 5.13.*

Since $h \in H \implies hx = hy$, *we just need to show* $(e^*h)x = (e^*h)y$ *if* $h \in H$. *We may assume* e *is apt for* x *and* y. *Let* γ_x *be the unique* Σ*-trajectory of* e *emanating from* x. *Let* $\rho : \Sigma_e \to \Sigma_e/\mathrm{kin}$ *be the kinship map. Then* γ_x *is a* Σ_e*-trajectory of* e *and* $\rho\gamma_x$ *is a* Σ_e/kin*-trajectory of* e *emanating from* $\rho x = \rho y$ *by Lemma 6.2. Hence* $\rho\gamma_x = \rho\gamma_y$, *where* γ_y *is defined like* γ_x. *Let* T *be the duration of* e. *If* $h \in H$ *and* $\hat{h}$ *is the corresponding element of* $\mathcal{O}_{\Sigma_e/\mathrm{kin}}$, *then*

$$(e^*h)x = h(xe) = h(\gamma_x T) = \hat{h}(\rho\gamma_x T) = (e^*h)y.$$

Thus x *and* y *are* Σ*-indistinguishable.*♠

Corollary 7.2 *Assume* Σ *is observable and that* $(X, \mathcal{U}(\Sigma))$ *is quasi-Hausdorff and locally full. Then distinct* Σ_e-kin *points are not isomobile.*♠

For C^∞ systems the situation is more complicated, because there is no way to infer equality of two C^∞ functions by just considering their derivatives at a single point. However note Example 7.9 involving C^∞ functions, at the end of this section.

Proposition 7.3 *Let* Σ *be a quasi-system,* x *a point of* Σ, e *an experiment. If* e *is apt for* x, *then* e *is apt for* x/kin.

Thus, in passing to Σ/kin, we can only increase aptness. To prove the proposition, we use the following lemma.

Lemma 7.4 *Let* e *be an experiment and* x *a point of the quasi-system* Σ. *If* e *has a unique trajectory emanating from* x, *it has a unique trajectory emanating from* x/kin.

Proof: *If* γ *is a trajectory of* e *emanating from* x *and* $\rho : X \to X/\mathrm{kin}$, *then* $\rho\gamma$ *is a trajectory of* e *emanating from* x/kin *(by Lemma 6.2). Conversely, let* δ *be any trajectory of* e *emanating from* x/kin *and let* T *be the duration of* e. *Define* $\gamma : [0, T] \to X$ *by letting* γt *to be any element of* $\rho^{-1}(\delta t)$ *if* $t \in (0, T]$ *and setting* $\gamma 0 = x$. *To show* δ *is unique, it will suffice to show that* γ *is a trajectory of* e, *for* $\rho\gamma = \delta$, *and the trajectory of* e *emanating from* x *is unique.*

For simplicity, we assume e *has only a single command* ω. *Let*

$$\psi = \rho^* : \mathcal{O}_{\Sigma/\mathrm{kin}} \to \mathcal{O}_\Sigma.$$

For $t \in [0, T_e]$, *we have*

$$\gamma^* D_\omega = \gamma^* \psi D'_\omega \psi^{-1} = (\rho\gamma)^* D'_\omega \psi^{-1}$$

$$= d/dt\ (\rho\gamma)^* \psi^{-1} = d/dt\ \gamma^* \rho^* \psi^{-1} = d/dt\ \gamma^*,$$

so γ *is a trajectory of* e *emanating from* x. ♠

Proof of Proposition 7.3
Assume e is apt for x, let $t \in [0, T]$, and apply Lemma 7.4 to the experiment $e_{\leq t}$. There is a unique trajectory of $e_{\leq t}$ emanating from x, hence also from x/kin. Thus e is apt for x/kin.♠

If P is a property of topological spaces, we shall say the quasi-system Σ *has property* P if its state space has property P. For instance, Σ is locally full if X is locally full. If Z is any open subset of the state space X, we can define a system $\Sigma_{|Z}$, by restricting everything to Z, as follows. If $\mathcal{D} = (D_\omega)_{\omega\in\Omega}$ is the Ω-field of Σ, we let $\mathcal{D}_{|Z}$ be $(D_{\omega|Z})_{\omega\in\Omega}$ where $D_{\omega|Z}$ is D_ω acting only on those $\varphi \in \mathcal{O}_X$ with $\text{dom}\varphi \subset Z$. We let $H_{|Z} = \{h + 0_Z : h \in H\}$. We let $\mathcal{O}_Z$ be $\mathcal{O}_{X|Z}$. Then

$$\Sigma_{|Z} = (Z, \mathcal{D}_{|Z}, H_{|Z}).$$

Proposition 7.5 *Let Σ be a quasi-system with the following properties:*

1. *Σ is selective,*
2. *Σ has trajectories,*
3. *Σ is locally full.*

Then Σ is a system.
Proof: *Let ω be a command, $x \in$ dom D_ω. We just need to establish that the local trajectory of D_ω emanating from x is unique. Let Z be an open neighborhood of x in the state space X of Σ such that $\Sigma_{|Z}$ is full. Since Z is Hausdorff by Proposition 4.8, the trajectory is unique by Theorem 5.13, as long as it stays in Z.* ♠

Fix a quasi-system $\Sigma = (X, \mathcal{D}, H)$ and assume that B is a function subuniverse of $\mathcal{O}_\Sigma$. Assume that $D_\omega B \subset B$ for every $\omega \in \Omega$ and that $H \subset B$. Let $D_{\omega|B}$ be the vector field on (X, B) obtained by restricting D_ω to B, and let $\mathcal{D}_{|B} = (D_{\omega|B})_{\omega\in\Omega}$. Regard the quasi-system $\Sigma_B = (X, \mathcal{D}_{|B}, H)$ with $\mathcal{O}_{\Sigma_B} = B$ and construct $\Sigma_B/B - \text{kin}$. We call it the *B-quotient* of Σ and denote by Σ/B-kin. From Proposition 7.5 we obtain the following.

Proposition 7.6 *Let Σ be a system and B as above. If B is locally full, then Σ/B-kin is a system.* ♠

In the example below, B is not locally full and the trajectories of the quotient quasi-system are not unique.

Example 7.7 *Let $X = \mathbb{R}^2$, $\mathcal{O}_X = \mathbb{A}_2$, $\mathcal{D} = \{D\}$, $H = \{h_1, h_2\}$, with $h_1 = z_1$, $h_2 = z_2$ the usual coordinate functions, but with* dom $h_2 = \{(x_1, x_2) : x_1 > 0\}$. *Let D be ∂_1. Let $B = \mathcal{C}(h_1, h_2)$. Then if $x \neq y$, x and y are B-kin iff $y_1 = x_1 \leq 0$. The quotient space X' looks like an open half-plane with an* *attached ray. We have unique trajectories of D and trajectories of experiments but that is not true for D'. Indeed $\gamma_s : [0, \infty) \to Y$ given by*

$$\gamma_s t = \begin{cases} [0] & \textit{if } t = 0 \\ (t, s) & \textit{if } t > 0 \end{cases}$$

are all trajectories of D' emanating from $[0] = \rho(0, 0)$.♠

It may happen that e is apt for x' in the quotient space and is not apt for any x such that $x' = \rho x$, even if trajectories of experiments are unique in both spaces.

Example 7.8 *Let $X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < x_2 < x_1 + 1\}$, $\mathcal{O}_X = (\mathbb{A}_2)_{|X}$, $B = \mathcal{C}(z_1)$ and $\mathcal{D} = \{D\}$, $D = \partial_1$, $H = \{z_1\}$. Then the quotient space X' looks like a real line and D' is equivalent to d/dt. We can compute the trajectory of any experiment e, emanating from any point $x' \in X'$, but the original system allows for only "short" experiments with duration less than 1. Notice that Σ is observable but not infinitesimally observable.* ♠

It is possible that a quasi-system Σ will be impossible to manage, whereas the quasi-system $\Sigma/\mathcal{U}(\Sigma) - \text{kin}$ will have much better properties. For example, we have noticed in Lemma 3.1 that if X is a C^∞ manifold and $\mathcal{O}_X$ consists of all $\varphi : X \to \mathbb{A}_0$ with $\text{dom}\varphi$ open and φ a C^∞ function on $\text{dom}\varphi$, then X is an analytic function universe. But we have not made this the basis for any of our examples, because trajectories of vector fields do not exist in general when $(X, \mathcal{O}_X)$ is regarded as an analytic function universe. For instance, if X is the open unit interval $(0, 1)$, Id_X is not a trajectory of d/dt (t the coordinate of X) because $\varphi \circ \text{Id}_X$ might not be analytic when φ is C^∞. The following example shows how this problem might dissapear for a system $\Sigma/\mathcal{U}(\Sigma) - \text{kin}$.

Example 7.9 *Let $X = \mathbb{R}^2$ and let $\mathcal{O}_X = C^\infty_{\mathbb{R}^2}$ be the set of all partially defined C^∞ functions on $\mathbb{R}^2$ considered as an analytic function universe. Let x and y denote the usual coordinate functions on $\mathbb{R}^2$. Define ω in $\mathcal{O}_X$ by*

$$\omega(x, y) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Let D be the vector field

$$D = x^2 \frac{\partial}{\partial x} + \frac{\partial}{\partial y}.$$

Let Σ be the quasi-system $(X, \{D\}, \{x, y, \omega\})$. We have

$$D\omega + 0/x = (2/x)\omega$$

and

$$D(1/x) = -1 + 0/x.$$

Therefore for $n \geq 0$

$$D^n\omega + 0/x = H_n\omega,$$

where H_n *is a polynomial of degree* n *in* $1/x$. *By continuity*

$$x^n D^n\omega = P_n\omega,$$

where P_n *is a polynomial of degree less or equal* n *in* x. *Define* ω_n *in* $\mathcal{O}_X$ *by* $\omega_0 = \omega$ *and* $x\omega_n = \omega_{n-1}$ *for* $n > 0$. *Then* $x, y, \omega_0, \omega_1, \omega_2, \ldots,$ *generate* $\mathcal{U}(\Sigma)$. *Observe that all the* ω_j *vanish along the* y*-axis.*

As mentioned above, trajectories of the vector field D *don't exist in the current setting of the system* Σ. *However, when we forget about the output structure and treat the dynamics as analytic, we easily find the trajectories of* D:

$$\gamma(t) = (0, b+t)$$

if γ *emanates from the point* $(0, b)$, *and*

$$\gamma(t) = (\frac{a}{1-at}, b+t)$$

if it emanates from (a, b) *with* $a \neq 0$.
They are still trajectories of D *if* $Y := \mathbb{R}^2$ *is regarded as a universe space by setting* $\mathcal{O}_Y = \mathcal{U}(\Sigma)$ *and* D *is identified with* $D_Y = D_{|\mathcal{O}_Y}$. *Then* Y *is selective and* $(Y, \{D_Y\}, \{x, y, \omega\})$ *is a system. It is isomorphic to* $\Sigma/\mathcal{U}(\Sigma) - \text{kin}$. ♠

8 Observable systems

We are now going to show how to construct observable systems from unobservable systems. For this, we shall have to consider a new condition on a quasi-system Σ which says in essence that the universe space structure of X_Σ is compatible with experiments.

Let Σ be a quasi-system, and let $\varphi \in \mathcal{O}_\Sigma$. We shall call φ *amenable* if $e^*\varphi \in \mathcal{O}_\Sigma$ for every experiment e. Also we say that Σ is *amenable* if every element of $\mathcal{O}_\Sigma$ is amenable. Because of the following result it is generally easy to determine when a quasi-system is amenable. We remark moreover that the quasi-systems that arise naturally always seem to be amenable. If e is any experiment for Σ, we define $\text{dom}e \subset X_\Sigma$ by

$$\text{dom}e = \{x \in X_\Sigma : e \text{ is apt for } x\}.$$

Lemma 8.1 *Let* Σ *be a quasi-system, and assume that* S *is a given set of generators of the function universe* $\mathcal{O}_\Sigma$. *The following are equivalent:*
1) Σ *is amenable.*
2) The following are both true for any experiment e:
a) dom$e$ *is open;*

b) $e^*S \subset \mathcal{C}(S)$.
Moreover b) $\Rightarrow$ *a) if* $\underline{0S} = 0_{\mathcal{O}_\Sigma}$.
Proof: *Assume 1) and let e be any experiment. Then* $\text{dom}e^*0 = \text{dom}e$, *so* $\text{dom}e$ *is open (we put* $0 = 0_{\mathcal{O}_\Sigma}$*). Obviously*

$$e^*S \subset \mathcal{C}(S) = \mathcal{O}_\Sigma.$$

Assume 2) and let e be any experiment. Let

$$W_e = \{\varphi \in \mathcal{O}_\Sigma : e^*\varphi \in \mathcal{O}_\Sigma\}.$$

Since $e^*0 \in 0\mathrm{IR}_X$ $(X = X_\Sigma)$ *and its domain is open,* $e^*0 \in \mathcal{O}_\Sigma$ *and* $0 \in W_e$. *If* $n > 0$, $\varphi_1, \ldots, \varphi_n \in W_e$ *and* $F \in \mathbb{A}_n$, *then* $F(\varphi) \in W_e$ *by Lemma 6.3. Also, by the same lemma,* $\underline{M} \in W_e$ *if* $M \in \mathcal{M}(W_e)$. *Therefore* W_e *is a function subuniverse of* $\mathcal{O}_\Sigma$ *and*

$$S \subset W_e \subset \mathcal{O}_\Sigma = \mathcal{C}(S)$$

so $W_e = \mathcal{O}_\Sigma$. *Therefore* $\varphi \in \mathcal{O}_\Sigma$ *implies* $e^*\varphi \in \mathcal{O}_\Sigma$ *and* Σ *is amenable.*
The final statement follows from $\text{dom}e = \text{dom}e^*0$, *where* $0 = 0_{\mathcal{O}_\Sigma}$, *and from*

$$e^*0 = e^*\underline{0S} = \underline{0e^*S},$$

whence b) implies that $\text{dom}e^*0$ *is the union of the open sets* $\text{dom}e^*s$, $s \in S$. ♠

By *experimental observation universe* of the quasi-system Σ, we mean the smallest function subuniverse U of $\mathcal{O}_\Sigma$ containing the observation structure H and such that $e^*U \subset U$ whenever $e \in E$ and $D_\omega U \subset U$ whenever $\omega \in \Omega$. We define it only if Σ is amenable, and we denote it by $\mathcal{E}(\Sigma)$.

Theorem 8.2 *Let* Σ *be a dynamic and amenable quasi-system. States x and y of* Σ *are* Σ*-indistinguishable iff x and y are* $\mathcal{E}(\Sigma) - \text{kin}$.
Proof: *Certainly if x and y are* $\mathcal{E}(\Sigma)$-kin, *they are* Σ*-indistinguishable since* e^*h *and* e^*0 *are in* $\mathcal{E}(\Sigma)$ *whenever* $e \in E$ *and* $h \in H$. *For the converse let* $\mathcal{T} = \mathcal{D} \cup E^*$, *where* E^* *consists of all operators* e^* *on* $\mathcal{O}_\Sigma$, $e \in E$, *and* $\mathcal{D} = \{D_\omega,\ \omega \in \Omega\}$. *Let* $S \subset \mathcal{E}(\Sigma)$ *consist of all* $T_1 \ldots T_r h$, *where* $r \geq 0$, $T_1, \ldots, T_r \in \mathcal{T}$, *and* $h \in H$. *Then* $H \subset S$, $D_\omega S \subset S$ *if* $\omega \in \Omega$, *and* $e^*S \subset S$ *if* $e \in E$.

Lemma 8.3 $\mathcal{E}(\Sigma) = \mathcal{C}(S)$.
Proof: *Evidently* $S \subset \mathcal{E}(\Sigma)$, *so* $\mathcal{C}(S) \subset \mathcal{E}(\Sigma)$. *If* $e \in E$, *then* $e^*S \subset S$ *so* $e^*\mathcal{C}(S) \subset \mathcal{C}(S)$ *by Lemmas 6.3 and 3.2. If* $\omega \in \Omega$, *then* $D_\omega \mathcal{C}(S) \subset \mathcal{C}(S)$ *by Lemma 5.5, since* $D_\omega(0_X) = \underline{0D_\omega H}$ *is in* $\mathcal{C}(S)$. *Thus* $\mathcal{C}(S)$ *is invariant under* $\mathcal{T}$, *so must equal* $\mathcal{E}(\Sigma)$.♠

Lemma 8.4 *No element of S can experimentally distinguish x from y if x and y are* Σ*-indistinguishable.*
Proof: *Let* $\varphi \in S$. *Write* $\varphi = T_1 \ldots T_r h$. *If* $r = 0$, $\varphi = h \in H$, *so it*

can't experimentally distinguish x from y. By induction on r, $\varphi = T\psi$ where $T = T_1,\ \psi = T_2 \dots T_r h$ and ψ cannot experimentally distinguish x from y. This implies $(e_1^*\psi)x = (e_1^*\psi)y$ whenever $e_1 \in E$, whence $((ee_1)^*\psi)x = ((ee_1)^*\psi)y$ whenever $e, e_1 \in E$. Thus if $T_1 = e_1^*,\ \varphi = T_1\psi$ can't experimentally distinguish x from y. If $T_1 = D_\omega,\ \omega \in \Omega$, then $\varphi = D_\omega\psi$ can't experimentally distinguish x from y by Lemma 6.7. ♠

From Lemma 8.4, $\varphi \in S \Rightarrow \varphi x = \varphi y$ if x and y are Σ-indistinguishable, so $\varphi x = \varphi y$ whenever $\varphi \in \mathcal{C}(S) = \mathcal{E}(\Sigma)$. Thus x and y are $\mathcal{E}(\Sigma)$-kin if they are Σ-indistinguishable. ♠

Let Σ be any quasi-system. We shall say that Σ *has backward trajectories* if whenever ω is a command of Σ, $-D_\omega$ has local trajectories. This means that if $x \in \mathrm{dom} D_\omega$, then there is an $\epsilon > 0$ and a trajectory $\gamma : (-\epsilon, 0] \to X$ of D_ω with $\gamma 0 = x$. Similarly, if $\mathcal{D} = \{D_\omega,\ \omega \in \Omega\}$ is the dynamics of Σ and local trajectories of $-\mathcal{D} = \{-D_\omega,\ \omega \in \Omega\}$ are unique, we say that *backwards local trajectories* of Σ *are unique*.

Let Σ be a quasi-system, and let $B = \mathcal{E}(\Sigma)$. Then we get the quotient quasi-system $\Sigma/B - \mathrm{kin}$. We denote it by $\Sigma' = (X', \mathcal{D}', H')$ and let $\pi : \Sigma \to \Sigma'$ be the canonical morphism. (One must be careful not to confuse π with the morphism ρ considered in Section 7.) We then have the following.

Theorem 8.5 *Let Σ be an amenable quasi-system which has forward and backward trajectories. Assume that trajectories of experiments of Σ and Σ' are unique, and that backward local trajectories of Σ' are unique. Then*
1) Σ' is an amenable and observable system;
2) whenever $x \in X$ and $e \in E$,

$$(F_x e)\pi^* = F_{\pi x} e.$$

3) $\mathcal{E}(\Sigma') = \mathcal{O}_{\Sigma'}$.
(In paraphrase, 2) says that every experiment will give identical results in Σ and Σ'.)
Proof: *Evidently Σ' is selective and dynamic, so it is a system. Before proving 3) and the rest of 1), we shall establish 2). For that, we note the following lemma. In it, D'_ω denotes the vector field of $\mathcal{O}_{\Sigma'}$, induced by $(D_\omega)_{|\mathcal{E}(\Sigma)}$ and the isomorphism $\pi^* : \mathcal{O}_{\Sigma'} \to \mathcal{E}(\Sigma)$.*

Lemma 8.6 *Let $\omega \in \Omega$, $x \in X$. Then D'_ω is defined at πx iff D_ω is defined at x.*
Proof: *From $\pi^* D'_\omega = D_\omega \pi^*$, where D_ω is to be interpreted in this formula as $(D_\omega)_{|\mathcal{E}(\Sigma)}$, we get*

$$(D'_\omega \varphi')(\pi x) = (D_\omega(\varphi'\pi))x,$$

for $x \in X$ and $\varphi' \in \mathcal{O}_{\Sigma'}$. Let $\varphi' = 0_{\mathcal{O}_{\Sigma'}}$, whence $\varphi'\pi = 0_{\mathcal{O}_\Sigma}$. Then D'_ω defined at πx means $(D'_\omega\varphi')(\pi x) = 0$ and D_ω defined at x means $(D_\omega(\varphi'\pi))x = 0$. ♠

For establishing 2) of Theorem 8.5, note that by Lemma 6.11, its equation holds if e *is apt for both* x *and* πx*, or if* e *is apt for neither. Therefore it will suffice to show that* e *is apt for* x *iff* e *is apt for* πx*. For* $T = 0$*, this follows immediately from Lemma 8.6, so we assume* $T > 0$*.*

Since trajectories of experiments in Σ' *are unique,* e *apt for* $x \Rightarrow$ e *is apt for* πx*. Therefore we assume* e *is apt for* πx*, but not for* x*, and derive a contradiction.*

Our assumptions imply that e *does not have any trajectory that emanate from* x*. Let* S *be the set of all* $t \in [0, T]$ *for which there is no trajectory of* $\mathrm{e}_{\leq t}$ *emanating from* x*, and observe that* $S \neq \emptyset$*, since* $T \in S$*. We let* $s = \mathrm{glb} S$*, and note that* $(s, T] \subset S$*.*

We first show $s \in S$*. If not, there is a trajectory* γ *of* $\mathrm{e}_{\leq s}$ *emanating from* x*. If* $s = T$*, the assumption that* e *is not apt for* x *is contradicted, so* $s < T$ *(under the assumption* $s \notin S$*). Choose* $\epsilon > 0$ *so that* e *has only one command* ω *on* $(s, s + \epsilon]$*. Then* ω *is the last command at* s*. Reducing* ϵ*, we get a trajectory* $\sigma : [s, s + \epsilon] \to X$ *of* D_ω *with* $\sigma s = \gamma s$*. Then* $\gamma\sigma$*, which is* γ *followed by* σ*, is a trajectory of* $\mathrm{e}_{\leq s+\epsilon}$*. But this can't be, since then* $s + \epsilon \notin S$*. Since* γ *allows us to construct* $\gamma\sigma$*, we see* γ *could not have existed in the first place, whence* $s \in S$*.*

Obviously when $\epsilon > 0$ *is sufficiently small, there is a trajectory of* $\mathrm{e}_{\leq \epsilon}$ *emanating from* x*, since by Lemma 8.6, if* ω *is any command of* e *at 0,* D_ω *is defined at* x*. Therefore* $0 < s \leq T$*. (The result* $s < T$ *was obtained with the disproved assumption* $s \notin S$*.)*

Let γ' *be the unique trajectory of* e *emanating from* $x' = \pi x$*. Let* $\gamma' s = y'$ *and pick* $y \in X$ *with* $\pi y = y'$*. (By "backtracking" from* y*, we shall be able to arrive at the desired contradiction that stems from assuming* e *apt for* πx *but not for* x*.) Pick* $\epsilon > 0$ *so that* e *restricted to* $(s - \epsilon, s)$ *has constant value* $\omega \in \Omega$*. Then* D'_ω *is defined at* y'*, so* D_ω *is defined at* y*. Since* Σ *has backward trajectories, after reducing* ϵ*, we have a trajectory* $\sigma : (-\epsilon, 0] \to X$ *of* D_ω *with* $\sigma 0 = y$*. Since* $\pi\sigma 0 = y' = \gamma'(s+0)$*, the uniqueness of backward local trajectories for* Σ' *implies that if* ϵ *is small enough, then*

$$\pi\sigma t = \gamma'(s + t), \; t \in (-\epsilon, 0].$$

Let $\eta \in (-\epsilon, 0)$*. By our choice of* s*, there is a unique trajectory* γ *of* $\mathrm{e}_{\leq s+\eta}$ *emanating from* x*. Then* $\pi\gamma = \gamma'_{|[0,s+\eta]}$*, so*

$$\pi\sigma\eta = \gamma'(s + \eta) = \pi\gamma(s + \eta)$$

by uniqueness of trajectories in Σ'*. Thus* $\sigma\eta$ *and* $\gamma(s+\eta)$ *are* Σ*-indistinguishable by Theorem 8.2, since evidently* Σ *is dynamic. The experiment* $(\omega, -\eta)$*, with trajectory* $t \mapsto \sigma(\eta + t)$*,* $0 \leq t \leq -\eta$ *emanating from* $\sigma\eta$*, is apt for* $\sigma\eta$*. It is not apt for* $\gamma(s + \eta)$*, since otherwise we could extend* γ *to a trajectory of* $\mathrm{e}_{\leq s}$ *contradicting* $s \in S$*. This contradiction establishes 2) of Theorem 8.5.*

Before we establish the rest of Theorem 8.5, let us note that if $x \in X$ *and*

$h \in \mathcal{O}_{\Sigma'}$, then for any experiment e, we have

$$(e^*\pi^*h)x = (F_x e)\pi^*h$$

and

$$(\pi^*e^*h)x = (F_{\pi x}e)h.$$

Thus the statement 2) of Theorem 8.5 can be written as

$$e^*\pi^* = \pi^*e^*,$$

but we must understand that e^* in $e^*\pi^*$ acts on $\mathcal{O}_\Sigma$ and that e^* in π^*e^* acts on $\mathcal{O}_{\Sigma'}$. The remaining parts of Theorem 8.5 will result from this last equation.

To show that Σ' is amenable, let $h' \in \mathcal{O}_{\Sigma'}$. Then $\pi^*h' \in \mathcal{E}(\Sigma)$ and for any experiment e

$$\pi^*e^*h' = e^*(\pi^*h')$$

is in $\mathcal{E}(\Sigma)$, so $e^*h' \in \mathcal{O}_{\Sigma'}$. Thus, since $\underline{0H'} = 0_{\mathcal{O}_{\Sigma'}}$, Σ' is amenable by Lemma 8.1.

To show Σ' is observable, assume $x', y' \in X'$ are Σ'-indistinguishable. Pick $x, y \in X$ with $\pi x = x'$, $\pi y = y'$. If $h' \in H'$ and e is any experiment, we can write $h = \pi^*h'$ and $h \in H$. Then

$$h(xe) = (e^*(\pi^*h'))x = (\pi^*e^*h')x =$$

$$(e^*h')x' = h'(x'e) = h(ye),$$

so x and y are Σ-indistinguishable. Thus x and y are $\mathcal{E}(\Sigma)$-kin *by Theorem 8.2, so $x' = y'$.*

Finally we show 3) of Theorem 8.5. We have observed that $e^*\pi^* = \pi^*e^*$, $e \in E$, and that $D_\omega\pi^* = \pi^*D'_\omega$, $\omega \in \Omega'$. Let S be the set used in proving Theorem 8.2 and S' the analogous set for the system Σ'. Our observations imply $\pi^*S' = S$. Also π^* induces an isomorphism $\mathcal{O}_{\Sigma'}$ with $\mathcal{E}(\Sigma)$. Since S generates $\mathcal{E}(\Sigma)$ and S' generates $\mathcal{E}(\Sigma')$, $\pi^*\mathcal{E}(\Sigma') = \mathcal{E}(\Sigma)$, so $\mathcal{E}(\Sigma') = \mathcal{O}_{\Sigma'}$. ♠

Corollary 8.7 *Let Σ be an amenable quasi-system that has forward and backward trajectories, and assume that trajectories of experiments for Σ are unique. Assume also that the universe space $(X, \mathcal{E}(\Sigma))$ is quasi-Hausdorff and locally full. Then the conclusions of Theorem 8.5 hold.* ♠

Corollary 8.8 *Let Σ be an amenable quasi-system that has forward and backward trajectories, and assume that trajectories of experiments for Σ are unique. Assume also that $\mathcal{E}(\Sigma)$ is global. Then the conclusions of Theorem 8.5 hold.* ♠

Example 8.9 *Let $X = \mathbb{R}^2$, $\mathcal{O}_X = \mathbb{A}_2$, $(D_\omega\varphi)(x_1, x_2) = \omega x_1 x_2 \partial_1\varphi + \omega\partial_2\varphi$, $\omega \in \mathbb{R}$ and H consists of two output functions: $h_1(x_1, x_2) = x_1$, defined for $x_1 < 1$ and $h_2(x_1, x_2) = x_1^2$ defined for $x_1 > 0$. Observe that $\mathcal{C}(H)$ is generated by the* *global coordinate function x_1. Then it is easy to show that $\mathcal{U}(\Sigma) = \mathcal{C}(x_1 x_2^i)_{i=0}^{\infty}$. The system may be represented by the set of equations*

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 u \\ \dot{x}_2 &= u \\ y_1 &= h_1(x_1, x_2) \\ y_2 &= h_2(x_1, x_2), \end{aligned}$$

where u is a control. Solving the equations we can compute trajectories of experiments. It is easy to see that $\mathcal{E}(\Sigma)$ is generated by global functions of the form $x_1 x_2^i \mathrm{e}^{x_2 t}$, where t is a real parameter. Obviously $\mathcal{U}(\Sigma) \subset \mathcal{E}(\Sigma)$. Neither of these universes separates points of the form $(0, x_2)$. Thus in the quotient space X', if $x_1 \neq 0$, the class $[(x_1, x_2)]$ consists of the single point (x_1, x_2), but $[(0,0)]$ is the whole x_2-axis. X' has a structure of universe space with $\mathcal{O}_{X'}$ isomorphic to $\mathcal{E}(\Sigma)$. In this case we can also use $\mathcal{U}(\Sigma)$ for $\mathcal{O}_{X'}$, since it separates the points of X'. But in neither case is $(X', \mathcal{O}_{X'})$ isomorphic to an analytic manifold. From Corollary 8.8, the quotient system Σ' is observable and the families of input-outputs maps for Σ and Σ' are the same. ♠

9 More examples

Example 9.1 *Let Σ be the system from Example 7.7. The observation universe $\mathcal{U}(\Sigma)$ is generated by h_1 and h_2. Note that $\mathcal{U}(\Sigma)$ does not distinguish points (x_1, x_2), (y_1, y_2) if $x_1 = y_1 \leq 0$. Let e_t be the experiment whose duration is t. Then $e_t^* h_2$ is defined for $x_1 > -t$. This means that using h_1 and $e_t^* h_2$ for t sufficiently large, we can distinguish any two distinct points of X. Thus $\mathcal{E}(\Sigma)$ separates points of X (actually $\mathcal{E}(\Sigma) = \mathbb{A}_2$, since $\mathrm{glb}\{e_t^* h_2\}_{t>0} = x_2$) and Σ is observable but not infinitesimally observable.*♠

Example 9.2 *Let $X = \mathbb{R}^2$, $\mathcal{O}_X = \mathbb{A}_2$, $\mathcal{D} = \{D\}$, $H = \{h_1, h_2\}$, where $D = x_1\partial_1 + x_2\partial_2$, $h_1(x_1, x_2) = x_2/x_1$, $h_2(x_1, x_2) = x_1/x_2$ with natural domains. Then $Dh_1 = 0h_1$, $Dh_2 = 0h_2$ so $\mathcal{U}(\Sigma) = \mathcal{C}(h_1, h_2)$. Moreover $\mathcal{E}(\Sigma) = \mathcal{U}(\Sigma)$. After passing to Σ', the origin forms an equivalence class. The other classes consist of straight lines passing 0, with the origin deleted. The vector field D' of Σ' is trivial and all trajectories are constant.*♠

Example 9.3 *Let $X = \mathbb{R}^2$, $\mathcal{O}_X = \mathbb{A}_2$, $\mathcal{D} = \{D\}$, where $D = (x_2 + \sin x_1)\partial_1 - x_1^3 x_2 \partial_2$, $H = x_1$. We find successively that $\mathcal{U}(\Sigma)$ contains $x_1, x_2 + \sin x_1, \sin x_1, x_2$ so $\mathcal{U}(\Sigma) = \mathbb{A}_2$. Hence, the system is observable by Corollary 6.8. It does not appear very easy to find trajectories of experiments here.* ♠

Example 9.4 *Let $\Sigma = (X, \mathcal{D}, H)$, where $X = \mathbb{R}^2$, $\mathcal{O}_X = \mathbb{A}_2$, $\mathcal{D} = \{D\}, H = \{x_1\}$, $D = \partial_1 + x_2^2\partial_2$. Two points of X are $\mathcal{C}(H) - \mathrm{kin}$ iff they have the same first coordinate. Observe that $\mathcal{C}(H)$ is stable under D and $X/\mathcal{C}(H) -$*

kin *is just* $(\mathbb{R}, \mathbb{A}_1)$. *Therefore* $(X, \mathcal{C}(H))$ *is quasi-Hausdorff and locally full. Obviously (or by Theorem 7.1)* $(a, b_1), (a, b_2)$ *will be* Σ*-distinguishable iff they are not isomobile. Any trajectory of* D *lies along the curve* $(t, 0),\ t \in \mathbb{R}$, *or a curve* $(t, \frac{1}{c-t}),\ t \in \mathbb{R}$, *for some* $c \in \mathbb{R}$. *A trajectory* $(t, \frac{1}{c-t})$ *will blow up at* $t = c$ *if it emanates from a point with positive second coordinate. We find that* $(a, b_1), (a, b_2),\ b_1 \neq b_2$ *are* Σ*-indistinguishable iff* $b_1 \leq 0$ *and* $b_2 \leq 0$. *If we let* $\mathcal{D} = \{D, -D\}$ *instead,* Σ *becomes observable.* ♠

Example 9.5 *Let* $X = \mathbb{R} \times S^1$ *where* $S^1 = \mathbb{R} \cup \{\infty\}$. *Let* $\mathcal{O}_X$ *be generated by* x_1, x_2 *on* $\mathbb{R}^2 \subset X$ *and* x_3 *defined by*

$$x_3(a, b) = b^{-1},\ a, b \in \mathbb{R}, b \neq 0;\ x_3(a, \infty) = 0,\ a \in \mathbb{R}.$$

We have a vector field D *on* X *(extending that of Example 9.4) given by* $Dx_1 = 1,\ Dx_2 = x_2^2,\ Dx_3 = -1$. *The system* $(X, \{\pm D\}, \{x_1\})$ *is not observable now.*♠

Lemma 9.6 *Let* W *be any open subset of* X *such that* dom $D_\omega \subset W$ *for every* $\omega \in \Omega$. *If* x *and* y *are distinguishable for the system* $\Sigma_{|W}$, *they are distinguishable for* Σ.

Proof *: Let* $e \in E$ *distinguish* x *and* y. *If, for example, the trajectory of* e *from* x *can be completed in* W, *whereas the one from* y *cannot, that will also be so for* X *because* dom $D_\omega \subset W$ *for every* $\omega \in \Omega$ *implies the trajectories of* e *are the same for either system. If* xe, ye *are both defined and there is* $h \in H$ *with* $h_{|W}(xe) \neq h_{|W}(ye)$, *then also* $h(xe) \neq h(ye)$ *since necessarily* $xe, ye \in W$. *Thus here also* x *and* y *are* Σ*-distinguishable.* ♠

Example 9.7 *Let* $\Sigma = (\mathbb{R}^2, \{D\}, \{x_1\})$, *where*

$$D = (\sin x_1 - \mathrm{e}^{x_2})\partial_1 + (1/\sin x_2)\partial_2$$

with domain $W = \mathbb{R}^2 \setminus (\mathbb{R} \times \mathbf{Z}\pi)$. *In succession we see that the functions* $x_1,\ Dx_1 = \sin x_1 - \mathrm{e}^{x_2} + 0_W,\ \sin x_1 - Dx_1 = \mathrm{e}^{x_2} + 0_W$ *are in* $\mathcal{U}(\Sigma)$. *At any point of* W, x_1 *and* e^{x_2} *have Jacobian* $\mathrm{e}^{x_2} \neq 0$, *so* $x_{1|W}$ *and* ${\mathrm{e}^{x_2}}_{|W}$ *generate the function universe* $\mathcal{O}_W$ *of all locally defined analytic functions on the open set* $W \subset \mathbb{R}^2$. *Thus* $\mathcal{U}(\Sigma_{|W}) = \mathcal{O}_W$, *so* $\Sigma_{|W}$ *is observable by Corollary 6.8. By Lemma 9.6, we see that distinct points of* W *are distinguishable for the system* Σ. *Also, since* $0_W \in \mathcal{U}(\Sigma)$, *any point of* $\mathbb{R} \times \mathbf{Z}\pi$ *can be distinguished from any point of* W *without even performing an experiment.*
No experiment can be performed on points of $\mathbb{R} \times \mathbf{Z}\pi$. *Now* $\mathcal{U}(\Sigma)$ *is generated, as a function universe, by* $x_1, Dx_1, D^2x_1, \ldots$, *but of these, only* x_1 *is defined at any point of* $\mathbb{R} \times \mathbf{Z}\pi$. *Thus points of* $\mathbb{R} \times \mathbf{Z}\pi$ *are* Σ*-indistinguishable iff they have the same first coordinate.*♠

Example 9.8 (Sussmann) *The system considered here has a* C^∞ *observation structure. Let* X *be a subset of* $\mathbb{R}^2$ *defined by*

$$(0, 2) \times (0, 3) \setminus [1, 2) \times [1, 2].$$

Let $\mathcal{O}_X = \{\varphi \in \mathbb{R}_X : \varphi \text{ is } C^\infty\}$. *Let* $\psi : (0,2) \to \mathbb{R}$ *be a* C^∞ *function equal* 0 *on* $(0,1]$ *and greater than* 0 *on* $(1,2)$. *Define* $h \in \mathcal{O}_X$ *by*

$$h(x_1, x_2) = \begin{cases} \psi(x_1) & \text{if } x_2 > 2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Sigma = (X, \{D\}, \{w\})$, *where* $D = \partial_1$. *In order to have existence of trajectories, we treat them as trajectories of the analytic vector field* ∂_1. *Consider points* $m = (1/2, 1/2)$, $p = (1/2, 3/2)$ *and* $q = (1/2, 5/2)$. *Observe that* m *and* p *are classically indistinguishable but they are not* Σ*-indistinguishable (one of the trajectories blows up at* $t = 1/2$ *while the other does not). The same happens to* p *and* q. *The points* m *and* q *can be classically distinguished, and hence, they can be distinguished in our sense. All the pairs considered are infinitesimally indistinguishable. This example shows that standard indistinguishability is not in general an equivalence relation even for globally defined systems.* ♠

Example 9.9 *Let* $\Sigma = (X, \{D\}, \{h\})$, *where* $X = T \setminus S$, T *is the two- dimensional torus,* $S = \{(x,y) : \sin^2 x + \sin^2 y \le \epsilon\}$, ϵ *is a small positive number,* h *is the function of Example 3.3 with domain* X, *and* $D = \partial_1 + \beta\partial_2$, $\beta \in \mathbb{R}, \beta \ne 0, 1$. *Regard* X *as a* C^ω *manifold in the usual way.*
Let

$$W = \{\sin 2x + 0h, \sin 2y + 0h, \cos 2x + 0h, \cos 2y + 0h\},$$

where the given functions are regarded as global elements of $\mathcal{O}_X$. *We claim that* $\mathcal{C}(W) = \mathcal{U}(\Sigma)$. *First observe that*

$$2Dh = \frac{\sin 2x + \beta \sin 2y}{h} \equiv \sin 2x + \beta \sin 2y + 0/h = \sin 2x + \beta \sin 2y + 0h,$$

since $h \ne 0$ *at any point of* X. *Two more differentiations and a little arithmetics show that* $W \subset \mathcal{U}(\Sigma)$. *Now* $h = \frac{1}{2}\sqrt{4 - 4\epsilon - (2\cos 2x + 0h + 2\cos 2y + 0h)}$ *shows* $h \in \mathcal{C}(W)$, *because the square root function is analytic on* $(0, \infty)$ *and the expression inside the square root is strictly positive on* X. *Since* $h \in \mathcal{C}(\mathcal{W}) \subset \mathcal{U}(\Sigma)$ *and* $\mathcal{C}(W)$ *is stable under* D, $\mathcal{C}(W) = \mathcal{U}(\Sigma)$. *Obviously* $\mathcal{C}(W) = \mathcal{O}_X$, *since* $\sin 2x$, $\cos 2x$ *generate* $\mathbb{A}_1$. *From Corollary 6.8 we see that* Σ *is observable.*♠

10 Remarks on future work

In this paper we developed a rigorous and unified approach to a theory of partially defined systems. We introduced the notion of universe space which served as a state space for such systems. As an example, we studied in detail the observability property and, in particular, passing to the quotient system with respect to the indistinguishability relation. Virtually all the existing theory of nonlinear systems could be rewritten using our general setting, but some areas seem to be especially appealing.

One of them is the controllability. Although there is still much to be done for globally defined systems, many problems of theoretical and practical interest could be stated and solved for partially defined systems. One of them is the relation between local and global reachability or local and global accessibility. As the first attempt one could consider a linear partially defined system and try to obtain criteria for global reachability. Obviously, besides algebraic conditions on the matrices of the system, there should appear conditions involving domains of vector fields and output functions. We hope that the calculus on universes will allow for elegant and simple characterization of those conditions.

Another area of potential study is realization theory. This is related to problems of finding different representations of control systems. One might be interested in globally defined models of partially defined systems or vice versa. An extensive study of transformations of partially defined systems should be an important element of this program. One of the questions that should be answered is, how more general are partially defined systems.

One should also try to introduce smaller classes of state spaces with more structure than general universe spaces. Such a class should contain differential manifolds and be closed under quotient formation. Another approach would be to define a kind of finiteness property in order to distinguish finite-dimensional universe spaces. We believe that one can obtain nice existence results for differential equations on such more concrete spaces.

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