Local observability of nonlinear systems

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Abstract

A necessary and sufficient condition for local observability of analytic systems is presented. For smooth systems this condition is only sufficient, but it is still weaker than the Hermann-Krener's rank condition. It is expressed with the language of ideals of germs of analytic or smooth functions and real radicals of such ideals. The condition may be practically checked in many cases.
Key words. Nonlinear control system. Local observability. Real radical. Germ. Rank condition.

1 Introduction

In 1977 Hermann and Krener [5] first proved the well-known rank condition for local observability of nonlinear systems. Even in the analytic case this condition is only sufficient. We show that the Hermann-Krener's condition is equivalent to maximality of a certain ideal in the ring of germs of analytic functions. Then we weaken up this condition replacing the ideal by its real radical. Our main result states that in the analytic case the system is locally observable (at a point) if and only if this radical is a maximal ideal. It may be proved that for smooth systems this new condition is sufficient but weaker than the Hermann-Krener's condition. The smooth case will be studied in a different paper. The condition of Hermann and Krener may be looked at as a first order condition for local observability. It fails, if only higher order terms are present. By considering the real radical we take those terms into account.

Though more complicated than Hermann-Krener's condition, our criterion is checkable in many cases. It requires more experience in playing with germs, ideals and radicals, but a skilled researcher can often check it quicker than the Hermann-Krener's condition.

There has been a growing interest in observability properties for systems described by polynomial or rational equations. The reader may want to consult e.g. [3] and the references cited there, in order to compare our analytic approach with more algebraic one.

2 The main result

Let Σ be a control system with output given by the equations

$$\dot{x} = f(x, u) \tag{1}$$

$$y = h(x). \tag{2}$$

We assume that $x(t)$ belongs to a C^ω n-dimensional manifold M (for example $\mathbb{R}^n$), $u(t)$ is an element of a set Ω, h is a C^ω map from M into $\mathbb{R}^r$, and for each $\omega \in \Omega$, $f_\omega := f(\cdot, \omega)$ is a C^ω vector field on M. We allow only piecewise constant controls u, so the equation (1), defining the dynamics, may be represented by the family of vector fields

$$\mathcal{D} = \{f_\omega : \omega \in \Omega\}.$$

Let $\gamma(t, x_0, u)$ denote the solution of (1) corresponding to the initial condition $\gamma(0) = x_0$ and the control u, and evaluated at time t. Let U be an open subset of M. By Σ_U we mean the restriction of Σ to U, i.e. restriction of $\mathcal{D}$, h and trajectories γ.

We say that $x_1, x_2 \in M$ are *indistinguishable* (with respect to Σ) if

$$h(\gamma(t, x_1, u)) = h(\gamma(t, x_2, u)), \tag{3}$$

for every control u and for every $t \geq 0$, for which both sides of (3) are defined. Otherwise, x_1 and x_2 are *distinguishable* (with respect to Σ). We say that Σ is *locally observable at* $x \in M$ if there is a neighborhood U of x such that for every $y \in U$, x and y are distinguishable. (In [5] this property is called weak observability.) We call Σ *strongly locally observable at* x, if there is a neighborhood U of x such that for every neighborhood V of x contained in U and every $y \in V$, x and y are distinguishable with respect to Σ_V (in particular the trajectories in (3) cannot leave V). (This property is called local weak observability in [5].)

By the *observation algebra* of the system Σ we mean the smallest subalgebra over $\mathbb{R}$ of $C^\omega(M)$ which contains all the components h_i of the map h and is closed under Lie derivatives with respect to vector fields f_ω, $\omega \in \Omega$. It is denoted by $\mathcal{H}(\Sigma)$, or simply $\mathcal{H}$ if the system is fixed. The algebra $\mathcal{H}$ contains in particular all constant functions.

Our aim now is to give a new necessary and sufficient condition for strong local observability. For $x \in M$, let $\mathcal{O}_x$ mean the algebra over $\mathbb{R}$ of germs of analytic functions at x. Define m_x to be the only maximal ideal of $\mathcal{O}_x$ ($\mathcal{O}_x$ is a local ring). It consists of all the germs that vanish at x.

Now, for an x in M, we define I_x – the ideal in $\mathcal{O}_x$ generated by germs at x of those functions from $\mathcal{H}$ which vanish at x. Obviously, I_x is contained in m_x.

Let us recall that the *real radical* of an ideal I in a commutative ring R, denoted by $\sqrt[\mathbb{R}]{I}$, is defined as the set of all elements $a \in R$ such that there are $m > 0, k \geq 0$ and elements $b_1, b_2, \ldots, b_k \in R$ such that $a^{2m} + b_1^2 + \ldots + b_k^2 \in I$. One can show that the real radical is an ideal in R (see e.g. [1]).

Now we can state the main result of this paper.

Theorem 2.1 *The following conditions are equivalent:*
(i) $\sqrt[\mathbb{R}]{I_x} = m_x$,
(ii) Σ *is strongly locally observable at* x,
(iii) Σ *is locally observable at* x.

We prove this result in Section 3. Below we present a few examples and some consequences of Theorem 2.1.

Example 2.2 *Let* $M = \mathbb{R}^2$ *and let the dynamics be trivial,* $\dot{x} = 0$. *Let* $r = 1$ *and* $h(x_1, x_2) = x_1^2 + x_2^2$. *One can easily see that* Σ *is locally observable at 0 and is not locally observable at any other point of* M. *Let us compute the radicals of Theorem 2.1. For* $x = 0$, $I_0 = (x_1^2 + x_2^2)$. *Observe that, from the definition of the real radical,* x_1 *and* x_2 *are in* $\sqrt[\mathbb{R}]{I_0}$, *so it is equal to* m_0. *On the other hand, if* $x = (c_1, c_2) \neq 0$ *then* I_x *is generated by* $x_1^2 - c_1^2 + x_2^2 - c_2^2$. *One can show (see e.g. [6]) that* I_x *is equal to its real radical. Hence, for* $x \neq 0$, $\sqrt[\mathbb{R}]{I_x} \neq m_x$. $\square$

Let $d\mathcal{H}(\Sigma, x)$ denote the linear space of differentials of all the elements from $\mathcal{H}(\Sigma)$ taken at point x.

Proposition 2.3 $\dim d\mathcal{H}(x) = n$ *iff* $I_x = m_x$.
Proof: *It is enough to prove the proposition for* $x = 0$ *in* $\mathbb{R}^n$.
Observe that $d\mathcal{H}(x) = dI_x(x)$ *for every* x. *Thus, if* $I_0 = m_0$, *then* $d\mathcal{H}(0)$ *contains all the differentials* dx_i *for* $i = 1, \ldots, n$.
On the other hand, the condition $\dim d\mathcal{H}(0) = n$ *implies that in a neighborhood* U *of* 0, *there are functions* $\varphi_1, \ldots, \varphi_n$ *whose germs belong to* I_0 *and the differentials at* 0, $d\varphi_i(0)$, *are linearly independent. We may write* $\varphi_i = \sum_j \Phi_{ij} x_j$ *for some analytic functions* Φ_{ij} *on* U *(sufficiently small). Then* $d\varphi_i(0) = \sum_j \Phi_{ij}(0) dx_j$. *This means that the matrix* $\Phi = (\Phi_{ij})$ *is invertible at* 0 *and then in some neighborhood of* 0. *Let* $\Psi = \Phi^{-1}$. *Then the germs of elements of* Ψ *are in* $\mathcal{O}_x$ *and* $dx_i = \sum_j \Psi_{ij} d\varphi_j$ *which means that* $\varphi_1, \ldots, \varphi_n$ *generate* m_0.
$\square$

The following theorem, first proved by R.Hermann and A.Krener, gives a simple sufficient condition for strong local observability.

Theorem 2.4 ([5]) *If* $\dim d\mathcal{H}(\Sigma, x) = n$, *then* Σ *is strongly locally observable at* x.
Proof: *If* $\dim d\mathcal{H}(x) = n$ *then* $I_x = m_x$ *by Proposition 2.3. Then also* $\sqrt[\mathbb{R}]{I_x} = m_x$. *This implies strong local observability at* x *by Theorem 2.1.* $\square$

Example 2.5 *Let $M = \mathbb{R}^3$ and the dynamics of Σ consist of one vector field $f(x_1, x_2) = -x_1\partial_1 + x_1^2\partial_2 + 3x_2^2\, \partial_3$, where $\partial_i = \frac{\partial}{\partial x_i}$. Let $h(x_1, x_2, x_3) = x_1^3 + 3x_2 + x_3$. Then $L_f h = -3x_1^3 + 3x_1^2 + 3x_2^2$, $L_f^2 h = 9x_1^3 - 6x_1^2 + 6x_2x_1^2$, and so on. One can see that all the differentials of functions from $\mathcal{H}$, besides h, evaluated at 0 are 0. So the sufficient condition of Theorem 2.4 is not satisfied. Observe that the germ at 0 of $L_f h = 3[x_1^2(1 - x_1) + x_2^2]$ belongs to I_0. Then, from the definition of real radical, x_2 and $x_1\sqrt{1 - x_1}$ belong to $\sqrt[\mathbb{R}]{I_0}$. Multiplying the last expression by the germ of $1/\sqrt{1 - x_1}$ we see that also $x_1 \in \sqrt[\mathbb{R}]{I_0}$ and finally, using h, we get that $x_3 \in \sqrt[\mathbb{R}]{I_0}$. By Theorem 2.1 this means that the system is strongly locally observable at 0.* $\square$

Remark 2.6 *The system in the preceding example is polynomial. A natural approach for such a system would be to consider the localization at x of the algebra of polynomials instead of $\mathcal{O}_x$. However the real radical of I_x would look differently when computed in this new ring. This means that as long as the standard topology on $\mathbb{R}^n$ is considered, one cannot take advantage of the algebraic structure of the system.* $\square$

3 Proof of the main result

The following fact is well known (see [5, 2]).

Lemma 3.1 *The points x_1 and x_2 are indistinguishable iff for every $\varphi \in \mathcal{H}$, $\varphi(x_1) = \varphi(x_2)$.*$\square$

Corollary 3.2 *Strong local observability at x and local observability at x are equivalent.* $\square$

For $x \in M$ and for an ideal $J \subset \mathcal{O}_x$, let $Z(J)$ be (the germ at x of) the zero-set of J. Since J is finitely generated ($\mathcal{O}_x$ is Noetherian), $Z(J)$ is well defined (see [4]). Let $I(Z(J))$ be the ideal in $\mathcal{O}_x$ of all germs of analytic functions that vanish on $Z(J)$.

Lemma 3.3 *The following conditions are equivalent:*
a) $Z(I_{x_0}) \neq \{x_0\}$;
b) arbitrarily close to x_0 there is x such that $\varphi(x_0) = \varphi(x)$ for every $\varphi \in \mathcal{H}$.
Proof: *a) $\Leftrightarrow$ arbitrarily close to x_0 there is x such that all representatives of germs $\varphi \in I_{x_0}$ (all defined in some neighborhood of x_0) are 0 at x $\Leftrightarrow$ all functions $\varphi \in \mathcal{H}$ take on the same values at x_0 and this x $\Leftrightarrow$ b).* $\square$

The following fact is essential in the proof of the main result.

Lemma 3.4 ([6]) *If J is an ideal in $\mathcal{O}_x$ then $I(Z(J)) = \sqrt[\mathbb{R}]{J}$.* $\square$

Proof of Theorem 2.1

Σ is not locally observable at x_0 iff arbitarily close to x_0 there is an x such that x_0 and x are indistinguishable. By Lemmas 3.1 and 3.3, the last statement is equivalent to the condition $Z(I_{x_0}) \neq \{x_0\}$, which in turn means that $I(Z(I_{x_0})) \neq I(\{x_0\})$. But $I(\{x_0\}) = m_{x_0}$, so from Lemma 3.4 the last inequality is equivalent to the condition $\sqrt[\mathbb{R}]{I_{x_0}} \neq m_{x_0}$. This and Corollary 3.2 give the equivalence of (i), (ii) and (iii). $\square$

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