Local observability and quasi-observers for nonlinear systems*

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Abstract

A new concept of quasi-observer for a nonlinear system is introduced. As observability of the system is not assumed, there are indistinguishable states in the state space. Therefore the output of the quasi-observer estimates one of the indistinguishable states of the system. Even if the original system is analytic, only continuity of the quasi-observer can be guaranteed. The lack of differentiability is caused mainly by a root function which often appears in the output. The main assumptions are local observability and Ascending Chain Property (introduced by Jouan and Gauthier). The main result says that under these assumptions the quasi-observer exists. The construction of the quasi-observer is similar as the construction of the observer given by Jouan and Gauthier. Several examples are provided.

1 Introduction

In recent papers [Ba1, Ba2, MB1, MB2] we presented necessary and sufficient conditions of local observability of continuous-time analytic systems. The complete proof for the most general case can be found in [Ba2]. The criterion is expressed in the language of real analytic geometry and the main mathematical tool we use is real radical of an ideal.

Jouan and Gauthier [JG] constructed an observer for a nonlinear system without control, assuming classical observability and finite multiplicity of a certain map associated with the system. The main idea was to find an observer that estimates the derivatives of the output and then, to recover an approximation of state from the output and its derivatives.

In this paper we relax the observability condition, assuming only local observability at every point. The main advantage of this change is that it is much easier to check local than global observability. In many cases Hermann-Krener rank condition holds (see [HK]), a sufficient condition for local observability. If not, one may try to verify more sophisticated necessary and sufficient conditions developed in [Ba1, Ba2, MB1, MB2]. When the system is locally observable, but not globally observable, it can have states that are indistinguishable. Local observability and analyticity guarantee however that the sets of indistinguishable states are discrete, i.e. they contain only isolated points. Thus in every compact set we can have only finitely many indistinguishable states.

All this means that, in general, under our assumptions, the true state cannot be recovered from the output data and the most we can get is a state indistinguishable from the true one. If we restrict the system to a compact subset of $\mathbb{R}^n$, then there are only finitely many such states.

To give some motivation of our approach let us consider the following system on $\mathbb{R}$:

$$\dot{x} = x, \quad y = x^2,$$

which is locally observable but not observable (the points x and $-x$ are indistinguishable for every $x \neq 0$). We could construct the quotient (observable) system by glueing up indistinguishable states, but then we would end up with a system on the closed half-line which is not a manifold. If we prefer to live in a more regular world, we must accept existence of indistinguishable points.

To recover x from the output y we take either $\sqrt{y}$ or $-\sqrt{y}$. If we are lucky we get the real state x, if not, we obtain $-x$, indistinguishable from x. This gives an idea of construction of a quasi-observer whose output estimates one of the indistinguishable states. In more complex systems it will be necessary to add a dynamics to the quasi-observer.

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2 Basic definitions

Let us consider the following system on $\mathbb{R}^n$:

$$\Sigma : \dot{x} = f(x), \quad y = h(x),$$

where f and h are analytic, $y \in \mathbb{R}$. Let $\gamma(t, x_0)$ denote the trajectory of Σ corresponding to the initial condition $x(0) = x_0$ and evaluated at time t.

Definition 2.1. Two points x_1 and x_2 in $\mathbb{R}^n$ are *indistinguishable* if

$$h(\gamma(t, x_1)) = h(\gamma(t, x_2))$$

for every $t \geq 0$ such that both sides of the equation are defined. Otherwise x_1 and x_2 are called *distinguishable.*

It is known that x_1 and x_2 are indistinguishable iff $\varphi(x_1) = \varphi(x_2)$ for every φ of the form $L_f^k h$, where L_f is the Lie derivative with respect to the vector field f.

Definition 2.2. We say that Σ is *observable* if any two distinct points are distinguishable.

Definition 2.3. Σ is *locally observable* at x_0 if there is a neighborhood U of x_0 such that for every $x \in U$, x and x_0 are distinguishable.

Definition 2.4. Σ is *stably locally observable* at x_0 if there is a neighborhood V of x_0 such that Σ is locally observable at every $x \in V$. Σ is *locally observable* if it is locally observable at every point, and this is equivalent to stable local observability at every point.

Now we are going to recall a few concepts from analytic geometry and real algebra.

Let $x \in \mathbb{R}^n$ and let $\mathcal{O}_x$ denote the algebra of germs at x of real analytic functions. It is known that $\mathcal{O}_x$ is Noetherian so every ideal of $\mathcal{O}_x$ is finitely generated. $\mathcal{O}_x$ is also a local ring with the only maximal ideal m_x consisting of all function-germs that vanish at x.

Let J be an ideal of $\mathcal{O}_x$. Then $\sqrt{J}$ denotes the *radical* of J, which consists of germs $\varphi \in \mathcal{O}_x$ such that $\varphi^k \in J$ for some $k \in \mathbb{N}$. Similarly we define the *real radical* of J, denoted by $\sqrt[\mathbb{R}]{J}$. A germ φ is in $\sqrt[\mathbb{R}]{J}$ if there are $m \in \mathbb{N}$, $k \geq 0$ and $\psi_1, \ldots, \psi_k \in \mathcal{O}_x$ such that

$$\varphi^{2m} + \psi_1^2 + \ldots + \psi_k^2 \in J.$$

Both ordinary and real radicals of a proper ideal J are proper ideals of $\mathcal{O}_x$ and the following inclusions hold:

$$J \subset \sqrt{J} \subset \sqrt[\mathbb{R}]{J} \subset m_x. \tag{2.1}$$

If G is a subset of $\mathcal{O}_x$ then $D(G)$ denotes the ideal in $\mathcal{O}_x$ generated by the Jacobians of n-tuples of elements from G.

For $\varphi \in \mathcal{O}_x$, $Z(\varphi)$ denotes the zero set-germ of the function-germ φ. If $J = (\varphi_1, \ldots, \varphi_k)$ then $Z(J) = Z(\varphi_1) \cap \ldots \cap Z(f_k)$ is well defined (see [Ba1, Ba2]). The following property is often used: if $J_1 \subset J_2$ then $Z(J_2) \subset Z(J_1)$.

Let $\mathcal{H}_x$ denote the subalgebra of $\mathcal{O}_x$ generated by the germs of the functions $L_f^k h, k \geq 0, i = 1, \ldots, r$. We define a sequence of ideals in the algebra $\mathcal{O}_x$ as follows. Let $I_x^{(0)} = (0)$ and $I_x^{(k+1)} = \sqrt[\mathbb{R}]{D(I_x^{(k)} \cup \mathcal{H}_x)}$. It can be shown that the sequence of ideals $I_x^{(k)}$ is increasing, so it must stabilize as the ring $\mathcal{O}_x$ is Noetherian. The following result was shown in [Ba2, MB2].

Theorem 2.5. *Σ is stably locally observable at x iff the sequence of ideals $I_x^{(k)}$ stabilizes at $\mathcal{O}_x$.* □

An equivalent criterion of stable local observability at x may be expressed using the set-germs $X_x^{(k)} := Z(I_x^{(k)})$. The sequence $X_x^{(k)}$ is decreasing and as a sequence of germs of analytic sets it must stabilize. Then Theorem 2.5 may be restated as follows: Σ is stably locally observable at x iff the sequence $X_x^{(k)}$ stabilizes at the empty set-germ.

Let us consider the map $\Phi_m : \mathbb{R}^n \to \mathbb{R}^m$ defined as follows:

$$\Phi_m(x) = (y_0, y_1, \ldots, y_{m-1}),$$

where $y_0 = h(x)$, $y_i = L_f^i h(x), i = 1, \ldots, m-1$. Let us fix $x^0 \in \mathbb{R}^n$ and define $A_m = (\Phi_m)^*(\mathcal{O}_{\Phi_m(x^0)})$. Then we have

$$A_1 \subset \cdots \subset A_m \subset A_{m+1} \subset \cdots \subset \mathcal{O}_{x^0}.$$

Jouan and Gauthier [JG] introduced an important property of Σ, called "Ascending Chain Property of order N" and denoted by ACP(N). Namely, Σ satisfies ACP(N) at x^0 if $A_m = A_N$, for $m \geq N$. We say that Σ satisfies ACP at x^0 if it satisfies ACP(N) for some N.

It can be shown that ACP(N) at x^0 is equivalent to the fact that in a neighborhood of $\Phi_n(x^0)$

$$y_m = \psi_m(y, y_1, \ldots, y_{N-1})$$

for some analytic function ψ_m and $m \geq N$ [JG].

Let I_x denote the ideal of $\mathcal{O}_x$ that consists of those function-germs from $\mathcal{H}_x$ that vanish at x. The main result of [Ba1] says the following:

Theorem 2.6. *System Σ is locally observable at x iff $\sqrt[\mathbb{R}]{I_x} = m_x$.* □

Using (2.1) we get a sufficient condition for local observability.

Corollary 2.7. *If $\sqrt{I_x} = m_x$ then Σ is locally observable at x.* □

Remark 2.8. The condition $\sqrt{I_x} = m_x$ may be interpreted as a criterion of local observability at x of the complexification of Σ. As in the complex case local observability is stable, this condition is also equivalent to stable local observability of the complexified system. On the other hand, the condition $\sqrt{I_x} = m_x$ is also equivalent to the property that for some m the map Φ_m has finite multiplicity (is finite) at x (see [JG, Ru, Na] for details).

Moreover, we have the following important property shown in [JG]:

Theorem 2.9. *If $\sqrt{I_x} = m_x$ then Σ satisfies ACP(N) at x for some $N \geq m$.* □

If $X = \mathbb{R}$, everything simplifies.

Proposition 2.10. *Let $X = \mathbb{R}$, then the following conditions are equivalent:*

(i) $\sqrt[\mathbb{R}]{I_x} = m_x$

(ii) ACP holds at x

(iii) local observability holds at x

3 Quasi-observers

Now we introduce the main concept of this paper.

Definition 3.1. A quasi-observer of system Σ is a system whose input is the output of system Σ and whose output is approaching any state indistinguishable from the current state of Σ, as the time increases.

We do not assume, as Jouan and Gauthier, that the map Φ_N is invertible for some N. This would mean observability of system Σ which is too strong a property. Instead we have to assume some kind of right invertibility of this map.

Condition A. There exists N and a continuous function $g : \text{im } \Phi_N \to \mathbb{R}^n$ such that $\Phi_N \circ g|_{\text{im } \Phi_N} = \text{id}\,|_{\text{im } \Phi_N}$.

Let K^0 be a compact subset of $X = \mathbb{R}^n$. We shall construct now a quasi-observer on K^0.

Theorem 3.2. *Let us assume that:*

1. *Σ is (stably) locally observable at every point of K^0,*
2. *ACP holds at each point of K^0,*
3. *Condition A holds for $\Phi_{N|K^0}$.*

Then there exists a continuous quasi-observer of system Σ on K^0.

Proof If the system Σ satisfies ACP at every point, then from [JG] there exists a continuous function φ such that

$$y_N = \varphi(y_0, \ldots, y_{N-1}).$$

Consider the system

$$S_{\Sigma,\theta,K^0} : \dot{Z} = (A - K_\theta C)Z + K_\theta y + b\varphi(Z), \qquad (3.1)$$

where $Z \in \mathbb{R}^N$, $Z = (Z_1, \ldots, Z_N)^T$, A is a $N \times N$ matrix, $A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}$, b is a $N \times 1$ matrix, $b = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$, C is a $1 \times N$ matrix, $C = (1, 0, \ldots, 0)$, $K_\theta = \Delta_\theta K$, K is a $N \times 1$ matrix such that $A - KC$ is Hurwitz and $\Delta_\theta = \text{diag}(\theta^i)$, $\theta \in \mathbb{R}_+$, $(\theta > 1)$. It was shown in [JG], that the state of system (3.1) exponentially estimates the first N derivatives of the output y of Σ (including $y = y_0$).

Since the condition A holds, then

$$x = g(y_0, \ldots, y_{N-1}).$$

We know that im $\Phi_N|_{K^0}$ is closed in X and by Tietze's theorem g can be extended to a continuous function $g^* : \mathbb{R}^N \to X$. Let us consider $\hat{x} = g^*(Z(t))$. $\hat{x}(t)$ is an estimation of $x(t)$ or an estimation of state indistinguishable from $x(t)$, so $\lim_{t\to 0} |\varphi(x(t)) - \varphi(\hat{x}(t))| = 0$, for every φ of the form $L_f^k h$, $k \geq 0$, whenever $\{x(t) : t \geq 0\} \subset K^0$. □

We can make stronger assumption than in Theorem 3.2 and then we obtain:

Theorem 3.3. *If $\sqrt{I_x} = m_x$ for each point $x \in X$ and condition A holds, then the quasi-observer exists on K^0.*

Proof Let $\sqrt{I_x} = m_x$ at each point $x \in X$. Since $\sqrt{I_x} \subseteq \sqrt[\mathbb{R}]{I_x} \subseteq m_x$, then $\sqrt[\mathbb{R}]{I_x} = m_x$ at each point $x \in X$. Hence the system is (stably) locally observable at each point x. Moreover, if $\sqrt{I_x} = m_x$ at each x, then ACP holds at every $x \in X$. Thus the assumptions of Theorem 3.2 are satisfied and the quasi-observer exists. □

4 Examples

Example 4.1. Let us consider the following system on $\mathbb{R}$

$$\dot{x} = x, \quad y = x^2$$

Since $\sqrt{I_x} = m_x$ at each point, hence the system is stably locally observable at every point and ACP(1) holds at every point. Then we can choose a matrix K, such that the matrix $A - KC$ is Hurwitz (for example $K = 2$) and the system

$$\dot{z} = (-\theta + 2)z + \theta y$$

gives an estimation of the output y for θ large enough. One may skip this part of the construction, as y is already given and further derivatives are not needed.

For this system condition A holds ($x = \pm\sqrt{y}$) and a function $g(z) = \sqrt{|z|}$ (or $g(z) = -\sqrt{|z|}$) is continuous and $\hat{x} = g(z)$ is an estimation of the state x of the system or an estimation of the equivalent state $(-x)$.

Example 4.2. Let $X = \mathbb{R}^2$. Consider the following system on X:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1$$
$$y = x_1^2 - x_2^2.$$

This system is stably locally observable and ACP holds at each point. In fact, we have $y_2 = -4y_0$ and then $\varphi(y_0, y_1) = -4y_0$. Then the system

$$S_{\Sigma,\theta,K^0}: \begin{array}{l} \dot{z}_1 = -2\theta z_1 + z_2 + 2\theta y \\ \dot{z}_2 = (-\theta^2 + 4)z_1 + \theta^2 y, \end{array}$$

(for $\theta > 1$ and large enough) gives an estimation of the output y and its first derivative (we took $K = (2,1)^T$). The output y is an input of the system S_{Σ,θ,K^0}. It may be shown that the map $\Phi_2(x) = (x_1^2 - x_2^2, 4x_1x_2)$ does not have a finite multiplicity. The map Φ_2 has a right inverse

$$g(y_0, y_1) = \left(\frac{y_1}{\sqrt{-2y_0 + \sqrt{4y_0^2 + y_1^2}}}, \sqrt{-\frac{y_0}{2} + \frac{\sqrt{4y_0^2 + y_1^2}}{4}}\right)$$

continuous besides the half-line $y_2 = 0$, $y_1 \geq 0$. However g cannot be extended onto the whole image of Φ_2, i.e. the whole plane, so Condition A is not satisfied. But $(\hat{x}_1, \hat{x}_2) = g(z_1, z_2)$ is still an estimation of the state $x \in X$ or $-x$, indistinguishable from x, as long as we keep away from the half-line.

Example 4.3. Let $X = \mathbb{R}^2$. The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \\ y &= x_1^2 - x_2^2 \end{aligned}$$

is stably locally observable and ACP(3) holds at each point, because the map

$$\Phi_3(x) = (x_1^2 - x_2^2, 2x_1x_2, 2x_2^2)$$

has finite multiplicity. We have $y_3 = 0$ and $\varphi(y_0, y_1, y_2) = 0$ for this system. Then the system

$$\dot{Z} = (A - K_\theta C)Z + K_\theta y$$

gives an estimation of the succesive derivatives of the output. The Condition A holds for this system. The function

$$g(y_0, y_1, y_2) = \left(\sqrt{y_0 + \frac{y_2}{2}}, \sqrt{\frac{y_2}{2}}\right)$$

is continuous and it can be extended to a continuous function on the whole $\mathbb{R}^3$:

$$g(z_1, z_2, z_3) = \left(\sqrt{|z + \frac{z_3}{2}|}, \sqrt{\frac{|z_3|}{2}}\right).$$

Then $(\hat{x}_1, \hat{x}_2) = g(z_1, z_2, z_3)$ gives an estimation of the state $x \in X$ or a state indistinguishable from x.

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