DYNAMIC FEEDBACK EQUIVALENCE OF NONLINEAR DISCRETE-TIME SYSTEMS

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Abstract. Dynamic feedback equivalence for discrete-time systems follows the continuous-time analog introduced by B.Jakubczyk. In dynamic feedback transformations current and past states and controls are used. We show that two systems are dynamically feedback equivalent iff their difference algebras are isomorphic. This is parallel to a theorem on equivalence of continuous-time systems. Another concept of anticipating feedback equivalence, using current and future states and controls, appears to be equivalent to the first one.

Key Words. Dynamic feedback equivalence, anticipating feedback equivalence, difference algebra, discrete-time nonlinear system.

1 INTRODUCTION

In the recent paper [5] a new definition of dynamic feedback equivalence of nonlinear continuous-time systems was presented. Though more abstract than previous attempts, this definition gives a clear and simple description of the feedback that depends on derivatives of states and controls.

In this paper we carry over these definitions and results to discrete-time systems. Instead of differential algebra associated with a continuous-time system, we use a difference algebra in which the derivation operator is replaced by a endomorphism of the algebra (called the difference operator). A similar language based on difference algebra was earlier used by M.Fliess [2, 3] in his papers concerning discrete-time systems. However the main tool we use in our paper, the difference algebra of a discrete-time system, seems to be a new concept. The reader may consult also [4, 6] which deal with discrete-time systems.

We give two definitions of feedback equivalence. In the first one, the feedback transformations exploit current and future states and controls. In the second one, current and past states and controls are used. This is important from a practical point of view. We show the two concepts of equivalence are equivalent.

The main result of this paper says that two discrete-time systems are dynamically feedback equivalent if and only if their difference algebras are isomorphic. The proofs of the results are omitted. They will appear elsewhere. But we present several lemmas which give an idea of the proof of the main result.

2 THE RESULTS

Let us consider the following discrete-time control system Σ

$$x(k+1) = f(x(k), u(k)), \tag{1}$$

where $k \in \mathbf{Z}$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$ and f is a map from $\mathbb{R}^n \times \mathbb{R}^m$ into $\mathbb{R}^n$. We shall assume that f is of class C^r, $r = -1, 0, 1, 2, \ldots, \infty, \omega$, where $r = -1$ means arbitrary, $r = 0$ continuous, and $r = \omega$ analytic.

By $J(n)$ we denote the set of sequences $X_r = (x(r), x(r+1), \ldots)$, where $r \in \mathbf{Z}$ and $x(i) \in \mathbb{R}^n$. Such a sequence may be treated as a function $x : [r, +\infty) \cap \mathbf{Z} \rightarrow \mathbb{R}^n$. Similarly U_r will denote a sequence in $J(m)$, and by $J(n,m)$ we shall mean the set of pairs (X_r, U_r), where $X_r \in J(n)$ and $U_r \in J(m)$. By a *trajectory* of Σ we mean any pair $(X_r, U_r) \in J(n,m)$ which satisfies (1) for $k \geq r$. This means that we consider experiments starting at any moment $r \in \mathbf{Z}$.

Let r_k denote the restriction operator, $r_k : J(n) \rightarrow J(n)$, $r_k(X_r) = X_r$ if $k \leq r$, and $r_k(X_r) = (x(k), x(k+1), \ldots)$ if $k > r$. The restriction operator works similarly on $J(m)$ and $J(n,m)$. If (X_r, U_r) is a trajectory, then also $(X_k, U_k) = r_k(X_r, U_r)$ is a trajectory for any $k \in \mathbf{Z}$. In general, trajectories cannot be extended backwards, but they are defined for all forward instants.

The *behavior* of Σ, denoted by $\mathcal{B}(\Sigma)$, is the set of all its trajectories. It is a subset of $J(n,m)$. $\mathcal{B}(\Sigma)$ is a union of $\mathcal{B}_r(\Sigma)$, $r \in \mathbf{Z}$, where $\mathcal{B}_r(\Sigma)$ consists of trajectories starting at time r.

Let $k \in \mathbf{Z}$ and $s_k : J(n) \rightarrow J(n)$ denote the shift operator defined by $s_k(X_r) = Y_{r+k}$ with $y(i) = x(i-k)$ for $i \geq r+k$. The shift operators work similarly on $J(m)$ and $J(n,m)$. Observe that the behavior $\mathcal{B}(\Sigma)$ is s_k-invariant for any k (the system (1) is stationary) and we have $s_k(\mathcal{B}_r(\Sigma)) = \mathcal{B}_{r+k}(\Sigma)$.

Let T be a map $\mathbb{R}^n \times J(m) \rightarrow J(n)$ defined by

$$T(x, U_r) := X_r = (x(r), x(r+1), \ldots),$$

where (X_r, U_r) is a trajectory of Σ and $x(r) = x$. It is clear that X_r is uniquely defined. We shall also need the projection map $P_{J(m)} : \mathbb{R}^n \times J(m) \rightarrow J(m)$, $P_{J(m)}(x, U_r) = U_r$.

We want the operator T to be left invertible, i.e. injective. To achieve this we shall assume the following condition:

Condition A. For every $x, y \in \mathbb{R}^n$ there is at most one u which satisfies the equation $y = f(x,u)$.

To get a C^k dependence of u on x and y in the above equation we shall need

Condition B. If the system is of class $C^k, k \geq 1$, then for any x and u the rank of the matrix

$$\frac{\partial f}{\partial u}(x,u)$$

is full (i.e. equal m).

For $k \geq 1$ we shall also assume

Condition C. The map $\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^n : (x,u) \mapsto (x, f(x,u))$ is proper, i.e. the inverse image of a compact set in $\mathbb{R}^n \times \mathbb{R}^n$ is a compact set in $\mathbb{R}^n \times \mathbb{R}^m$.

We are going to study real functions and maps on $J(n)$ or similar spaces. We shall always assume that such a function φ is shift invariant, i.e. $\varphi(s_k(X_r)) = \varphi(X_r)$, so it is enough to define it only for sequences X_0. Moreover each such function will depend only on a finite number of elements $x(0), x(1), \ldots, x(i)$ of the sequence X_0. However i will not be fixed and will depend on the function φ. Regularity of such a function is just the regularity of a function of several real variables.

Let ϕ be a map from $J(\tilde{n})$ to $\mathbb{R}^n$. By the *extension* of ϕ we mean the map $\Phi : J(\tilde{n}) \rightarrow J(n)$ defined by

$$\Phi(\tilde{X}_r) = X_r, \quad x(i) = \phi(r_i(X_r)), \ i \geq r.$$

One easily shows that the extension map Φ commutes with the shifts s_k, i.e. $s_k(\Phi(\tilde{X}_r)) = \Phi(s_k(\tilde{X}_r))$. This follows from the formula $s_k r_{i-k} = r_i s_k$. One similarly defines the extension of a map $\psi : J(\tilde{n}, m) \rightarrow \mathbb{R}^m$.

Let us consider two systems

$$\Sigma : \ x(k+1) = f(x(k), u(k))$$

and

$$\tilde{\Sigma} : \ \tilde{x}(k+1) = \tilde{f}(\tilde{x}(k), \tilde{u}(k))$$

with $x(k) \in \mathbb{R}^n$, $\tilde{x}(k) \in \mathbb{R}^{\tilde{n}}$, $u(k), \tilde{u}(k) \in \mathbb{R}^m$, and the maps

$$\phi : J(\tilde{n}) \rightarrow \mathbb{R}^n, \quad \psi : J(\tilde{n}, m) \rightarrow \mathbb{R}^m \tag{2}$$

and

$$\tilde{\phi} : J(n) \rightarrow \mathbb{R}^{\tilde{n}}, \quad \tilde{\psi} : J(n,m) \rightarrow \mathbb{R}^m. \tag{3}$$

The maps $\phi, \psi, \tilde{\phi}$ and $\tilde{\psi}$ represent *anticipating feedback transformations*

$$x(k) = \phi(\tilde{X}_k), \ u(k) = \psi(\tilde{X}_k, \tilde{U}_k)$$

and

$$\tilde{x}(k) = \tilde{\phi}(X_k), \ \tilde{u}(k) = \tilde{\psi}(X_k, U_k).$$

The extensions $\Phi, \Psi, \tilde{\Phi}$ and $\tilde{\Psi}$ define maps:

$$\begin{aligned} \chi = (\Phi, \Psi) \quad &: \quad J(\tilde{n}, m) \rightarrow J(n, m) \\ &: \quad (\tilde{X}_r, \tilde{U}_r) \mapsto (\Phi(\tilde{X}_r), \Psi(\tilde{X}_r, \tilde{U}_r)), \end{aligned}$$

and

$$\begin{aligned} \tilde{\chi} = (\tilde{\Phi}, \tilde{\Psi}) \quad &: \quad J(n, m) \rightarrow J(\tilde{n}, m) \\ &: \quad (X_r, U_r) \mapsto (\tilde{\Phi}(X_r), \tilde{\Psi}(X_r, U_r)). \end{aligned}$$

We say that systems Σ and $\tilde{\Sigma}$ are *anticipating feedback equivalent* if there exist maps (2) and (3) such that $\chi(\mathcal{B}(\tilde{\Sigma})) = \mathcal{B}(\Sigma)$, $\tilde{\chi}(\mathcal{B}(\Sigma)) = \mathcal{B}(\tilde{\Sigma})$, and χ and $\tilde{\chi}$ are mutually inverse on behaviors.

The anticipating feedback defined above may not be of practical use, since we have to use future values of states and controls. The concept of dynamic feedback introduced below seems to be more suitable for applications.

We say that a function $\varphi : J(n) \rightarrow \mathbb{R}$ *is of order less or equal* p, if φ depends only on the first $p+1$ elements of a sequence X_r. A similar definition holds for φ defined on $J(n, m)$. We say that a pair (ϕ, ψ) in (2) *is of order less or equal* p if all the components of ϕ and ψ have this property.

We say that systems Σ and $\tilde{\Sigma}$ are *dynamically feedback equivalent* if there exist maps (ϕ, ψ) given by (2) of order less or equal p, and $(\tilde{\phi}, \tilde{\psi}$ given by (3) of order less or equal q, such that the maps $\chi = (\Phi, \Psi)$ and $\tilde{\chi} = (\tilde{\Phi}, \tilde{\Psi})$ have the following properties:

$$(\chi s_p)(\mathcal{B}(\tilde{\Sigma}) = \mathcal{B}(\Sigma), \ (\tilde{\chi} s_q)(\mathcal{B}(\Sigma)) = \mathcal{B}(\tilde{\Sigma}),$$

and

$$\chi s_p \tilde{\chi} s_q = s_{p+q}, \ \tilde{\chi} s_q \chi s_p = s_{p+q}$$

when restricted to behaviors $\mathcal{B}(\tilde{\Sigma})$ and $\mathcal{B}(\Sigma)$ respectively.

The maps $\phi, \psi, \tilde{\phi}$ and $\tilde{\psi}$ represent *dynamic feedback transformations*

$$\begin{aligned} x(k) &= \phi(s_p(\tilde{X}_k)) \\ &= \phi(\tilde{x}(k-p), \ldots, \tilde{x}(k), \\ u(k) &= \psi(s_p(\tilde{X}_k, \tilde{U}_k)) \\ &= \psi(\tilde{x}(k-p), \ldots, \tilde{x}(k), \tilde{u}(k-p), \ldots, \tilde{u}(k)) \end{aligned}$$

and similarly the other way. These transformations depend on past and current states and controls.

Note that, using the system equation, X_r can be (uniquely) expressed as a function of $x(r)$ and U_r. This means that, when restricting to trajectories, we can choose $\tilde{\psi}$ as a function of $x(r)$, $U(r)$ only.

The following result says that the two concepts of feedback equivalence are close to each other.

Proposition 2.1 *The systems Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent iff they are anticipating feedback equivalent.*

By a *difference algebra* (see [Co,Fl]) we mean an algebra A over $\mathbb{R}$ together with a homomorphism d of A into itself. Let (A_1, d_1) and (A_2, d_2) be two difference algebras. A *homomorphism* of the difference algebras from A_1 into A_2 is a homomorphism of algebras $\tau : A_1 \rightarrow A_2$ such that $d_2 \circ \tau = \tau \circ d_1$. An *isomorphism* of the difference algebras A_1 and A_2 is a homomorphism which is a bijective map.

Let $A(n, m)$ denote the algebra of all functions

$$\varphi : \mathbb{R}^n \times J(m) \rightarrow \mathbb{R}$$

(depending only on a finite number of elements in $U \in J(m)$ and invariant with respects to the shifts s_k). We will assume that the functions φ are of the same class as the map f. Let us now consider a system Σ, described by (1). Define the difference operator d_Σ associated with Σ by

$$(d_\Sigma \varphi)(x, U_r) := \varphi(f(x, u(r)), r_{r+1} U_r). \qquad (4)$$

One can show that d_Σ is an endomorphism of $A(n, m)$, so the algebra $A(n, m)$ together with the d_Σ is a difference algebra called the *difference algebra of system* Σ and denoted by A_Σ.

The main result of this paper says the following

Theorem 2.2 *Systems Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent iff their difference algebras A_Σ and $A_{\tilde{\Sigma}}$ are isomorphic.*

The proof of the above theorem rests on several lemmas given below. Let $\eta = (\eta^1, \eta^2)$, where $\eta^1 = (\eta_1, \ldots, \eta_n)$, $\eta^2 = (\eta_{n+1}, \ldots, \eta_{n+m})$ and $\eta_s \in A(\tilde{n}, m), s = 1, 2, \ldots, n+m$.

Lemma 2.3 *If two systems Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent via transformations (2), (3), and η^1 and η^2 are defined by $\eta^1 = \phi \circ T$, $\eta^2 = \psi \circ (T, P_{J(m)})$, then $d_{\tilde{\Sigma}} \eta^1 = f \circ \eta$.*

Let us define a map

$$\hat{\eta} : \mathbb{R}^{\tilde{n}} \times J(m) \rightarrow \mathbb{R}^n \times J(m)$$

by $\hat{\eta} = (\eta^1, \eta^2, (d^j_{\tilde{\Sigma}} \eta^2)_{j=1,2,\ldots})$. Consider the pullback $\hat{\eta}^*(\varphi) = \varphi \circ \hat{\eta}$, $\hat{\eta}^* : A(n, m) \rightarrow A(\tilde{n}, m)$.

Lemma 2.4 *If the assumptions of Lemma 2.3 are satisfied then the map $\hat{\eta}^*$ is a homomorphism of difference algebras A_Σ and $A_{\tilde{\Sigma}}$.*

Let $\tau : A_\Sigma \rightarrow A_{\tilde{\Sigma}}$ be a homomorphism of difference algebras. Let $x^i : \mathbb{R}^n \times J(m) \rightarrow \mathbb{R}$, $u^j_k : \mathbb{R}^n \times J(m) \rightarrow \mathbb{R}$ be the coordinate functions

$$x^i(x, U_r) = x_i, i = 1, \dots, n, \tag{5}$$

$$u^j_k(x, U_r) = u(r+k)_j, j = 1, \dots, m. \tag{6}$$

Lemma 2.5 *For any homomorphism $\tau : A(n, m) \rightarrow A(\tilde{n}, m)$ of difference algebras there exists a unique map η such that $\tau = \hat{\eta}^*$. Then $\tau(x^i) = \eta_i$, $i = 1, \dots, n$ and $\tau(u^j_0) = \eta_{n+j}$, $j = 1, \dots, m$.*

The last lemma gives an idea of a dynamic feedback which realizes the equivalence between Σ and $\tilde{\Sigma}$.

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