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COMPUTATIONAL COMPLEXITY OF OBSERVABILITY PROBLEMS

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Abstract

Observability of nonlinear analytic systems may be characterized by several properties. To check each of them one needs to compute several Lie derivatives of the output with respect to the vector field of the system. How many derivatives must be computed depends heavily on the problem.
Keywords: observability; nonlinear system; Lie derivative; computational complexity; real radical.

1 Introduction

Observability of nonlinear systems may be defined in many ways. Even local observability which is emphasized in this paper may have stronger or weaker versions (see [Ba1, Ba3, HK, MB2]). Most of these properties may be checked in practice. However the amount of computations needed to check the properties is different for each property.

Let us consider the system Σ:

$$\dot{x} = f(x) \tag{1.1}$$

$$y = h(x), \tag{1.2}$$

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where $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, f and h are analytic. We study here only systems with scalar outputs and without controls to simplify the language and notation.

Let $x(t, x_0)$ denote the solution of (1.1) corresponding to the initial condition $x(0) = x_0$ and evaluated at time t. Points x_1 and x_2 in $\mathbb{R}^n$ are called *indistinguishable* if $h(x(t, x_1)) = h(x(t, x_2))$ for all $t \geq 0$ for which both sides of the equation are defined. Otherwise x_1 and x_2 are distinguishable.

For $\varphi : \mathbb{R}^n \to \mathbb{R}$ let $L_f \varphi = \nabla\varphi \cdot f$ denote the Lie derivative of φ with respect to the vector field f. Let $\mathcal{H} = \{h, L_f h, L_f^2 h, \ldots\}$. This family of functions is the main tool in characterizing the indistinguishability relation.

Theorem 1.1 ([HK]). *Points x_1 and x_2 are indistinguishable iff $L_f^k h(x_1) = L_f^k h(x_2)$ for $k \geq 0$.*

Let us recall several different observability properties of nonlinear systems.

Definition 1.2.

- Σ is *globally observable* (GO) if any two indistinguishable points are equal.
- Σ is *locally observable* at x_0 $(LO(x_0))$ if there is a nbhd U of x_0 such that for every $x \in U$ distinct from x_0, x and x_0 are distinguishable.
- Σ is *locally observable* (LO) if it is locally observable at any $x_0 \in \mathbb{R}^n$.
- Σ is *stably locally observable* at x_0 $(SLO(x_0))$ if there is a nbhd V of x_0 such that Σ is $LO(x)$ for every $x \in V$.
- Σ satisfies *Hermann-Krener rank condition* at x_0 $(HK(x_0))$ if $\dim \operatorname{span} d\mathcal{H}(x_0) = n$.
- Σ satisfies *generic Hermann-Krener rank condition* $(GeHK)$ if $HK(x)$ holds almost everywhere in $\mathbb{R}^n$ (i.e. in an open dense subset of $\mathbb{R}^n$).
- The family $\mathcal{H}$ is *injective at* x_0 $(IN(x_0))$ if there is a neighborhood W of x_0 such that for every $x_1, x_2 \in W$: the condition $\varphi(x_1) = \varphi(x_2)$ for every $\varphi \in \mathcal{H}$ implies $x_1 = x_2$.

Observe that stable local observability at all points of $\mathbb{R}^n$ is equivalent to local observability. All the properties of Σ in Definition 1.2 are in fact properties of the set $\mathcal{H}$, so we often say that $\mathcal{H}$ is locally observable or $\mathcal{H}$ satisfies the Hermann-Krener rank condition at x_0.

2 Problem statement

The relations between various observability properties given in Definition 1.2 are described in the following proposition. None of the implications can be reversed. No other implications hold besides those implied by transitivity.

Proposition 2.1. *The following implications hold:*

$$\begin{array}{ccccc} & & HK(x_0) & & \\ & & \Downarrow & & \\ GO & \Rightarrow & IN(x_0) & & \\ \Downarrow & & \Downarrow & & \\ LO & \Rightarrow & SLO(x_0) & \Rightarrow & GeHK \\ & & \Downarrow & & \\ & & LO(x_0). & & \end{array}$$

Let us state now the following

Problem. How many Lie derivatives $L_f^k h$ does one have to compute to check each of the properties?

Proposition 2.2. *To check global observability, in general, all Lie derivatives are needed.*

Example 2.3. Let

$$\begin{aligned} \dot{x} &= 1 \\ y &= h(x) \end{aligned}$$

and $h : \mathbb{R} \to \mathbb{R}$ is an analytic function with zeros $x_k = k$, $k \in \mathbb{N}$, x_k has multiplicity k (one can construct such a function using the Weierstrass product). Then $L_f^k h = h^{(k)}$. To distinguish k and $k+1$ one has to compute kth derivative of h.

Let $HK_k(x_0)$ denote the following condition:

$$\dim \operatorname{span}\{dh(x_0), dL_f h(x_0), \ldots, dL_f^{k-1} h(x_0)\} = n.$$

This condition means that the Hermann-Krener rank condition is achieved after taking $k-1$ Lie derivatives of the output.

Proposition 2.4. *For each x_0 there is $k \geq n$ such that:*

$$HK(x_0) \Leftrightarrow HK_k(x_0).$$

Proposition 2.4 says that locally only finite number of Lie derivatives is needed to check the Hermann-Krener rank condition. However this number may be arbitrarily large.

Example 2.5. Let

$$\begin{aligned} \dot{x} &= 1 \\ y &= x^m \end{aligned}$$

Then $HK(0) \Leftrightarrow HK_m(0)$ and $HK(x_0) \Leftrightarrow HK_1(x_0)$ for $x_0 \neq 0$.

For polynomial systems there should exist an universal k depending only on n and the maximal degree of the polynomials in the system as it is for the accessibility problem [Ga]. However no such estimates are known in general case.

For the generic Hermann-Krener rank condition the problem has a simple solution.

Let $GeHK_k$ mean $HK_k(x_0)$ for almost every $x_0 \in \mathbb{R}^n$.

Theorem 2.6. *For any system Σ: $GeHK \Leftrightarrow GeHK_n$.*

Thus the set of points for which we need more than $n-1$ Lie derivatives is small.

To check injectivity at x_0 we may need more than $n-1$ Lie derivatives.

Example 2.7. Let

$$\begin{aligned} \dot{x} &= 1 \\ y &= x^2. \end{aligned}$$

To show injectivity at $x_0 = 0$ we need $h(x) = x^2$ and $L_f h(x) = 2x$.

3 Radicals

Let $\mathcal{O}_{x_0}$ denote the ring of germs of analytic functions at x_0. Let I_{x_0} be the ideal in $\mathcal{O}_{x_0}$ generated by functions $L_f^k h - L_f^k h(x_0)$, $k \geq 0$.

Definition 3.1. Let I be an ideal of a commutative ring P.

- $a \in \sqrt{I}$ (*radical* of I) if $a^k \in I$ for some $k \in \mathbb{N}$.
- $a \in \sqrt[\mathbb{R}]{I}$ (*real radical* of I) if $a^{2m} + b_1^2 + \ldots + b_k^2 \in I$ for some $m \in \mathbb{N}$, $k \geq 0$ and $b_1, \ldots, b_k \in P$.
- Ideal I is *real* if it is equal to its real radical.

Let $m_{x_0} = (x_1 - x_{01}, \ldots, x_n - x_{0n})$ be the maximal ideal of $\mathcal{O}_{x_0}$.

Proposition 3.2. *The following inclusions hold:*

$$I_{x_0} \subset \sqrt{I_{x_0}} \subset \sqrt[\mathbb{R}]{I_{x_0}} \subset m_{x_0}.$$

The radicals allow to extend the array of implications from Proposition 2.1

Theorem 3.3. *The following implications hold*

$$
\begin{array}{ccccc}
 & & HK(x_0) & \Leftrightarrow & I_{x_0} = m_{x_0} \\
 & & \Downarrow & & \\
GO & \Rightarrow & IN(x_0) & & \\
 & & \Downarrow & & \\
\Downarrow & & \sqrt{I_{x_0}} = m_{x_0} & & \\
 & & \Downarrow & & \\
LO & \Rightarrow & SLO(x_0) & \Rightarrow & GeHK \\
 & & \Downarrow & & \\
 & & LO(x_0) & \Leftrightarrow & \sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}.
\end{array}
$$

Let $I_{x_0|k}$ be the ideal of $\mathcal{O}_{x_0}$ generated by functions:

$$h - h(x_0), \ldots, L_f^{k-1} h - L_f^{k-1} h(x_0).$$

Proposition 3.4. *Assume that the ideals $I_{x_0|k}$ are real and prime for $k = 1, \ldots, n$. Then to check local observability at x_0 it is enough to compute $n-1$ first Lie derivatives of h.*

If the assumptions in Proposition 3.4 are not satisfied, we may need more Lie derivatives.

Example 3.5. Let $h(x_1, x_2) = x_1^2$ and $f(x_1, x_2) = (x_2, 0)^T$. Then to establish local observability at 0 we need h, $L_f h(x_1, x_2) = 2x_1x_2$ and $L_f^2 h(x_1, x_2) = 2x_2^2$.

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