INPUT-OUTPUT AND TRANSFER EQUIVALENCE OF LINEAR CONTROL SYSTEMS ON TIME SCALES

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Abstract. Transfer matrix and transfer equivalence for linear control systems on time scales are introduced. These concepts generalize continuous-time and discrete-time versions. Necessary and sufficient conditions for transfer equivalence are presented. As the main tool an extension of the Laplace transform to functions defined on time scales is used.

Key Words. Time scale, generalized Laplace transform, transfer matrix, input-output equivalence, transfer equivalence

1. INTRODUCTION

The transfer equivalence for control systems has been studied both in the continuous-time and discrete-time cases. Especially for linear systems the definitions, properties and results are very similar or even identical, for example [11, 3, 10].

The language of time scales, created in 1988 by Stefan Hilger [7] seems to be an ideal tool to unify the theories of continuous and discrete-time systems. One of the main concepts is the delta derivative, which is a generalization of ordinary (time) derivative. If the time scale is the real line, we get ordinary derivative. In the case of integer numbers, delta derivative of a function is the difference of its values at subsequent points. Thus differential equations as well difference equations are naturally accommodated into the theory. An inverse operation to differentiation, i.e. integration has been also extended into the time scale domain.

The transfer matrix of continuous-time and discrete-time control system is defined via the Laplace transform and the $\mathcal{Z}$-transform of the input-output equation, respectively. Using time scales allows to unite both transforms into one concept - the Laplace transform on time scale. But it allows much mare; besides standard discrete- and continuous-time systems this approach allows to study several other cases as, for examples time scales based on Cantor set, harmonic numbers, etc. Another approach to unify the Laplace transform and $\mathcal{Z}$-transform, but not contains so many cases, was introduced in [5].

The main goal of this paper is to study the transfer equivalence of linear control systems, described by input-output polynomial (in delta derivative operation) equations. The paper is organized as follows. In section 2 we recall the calculus on time scales. Section 3 presents the concept of Laplace transform on a time scale and Section 4 defines transfer equivalence for linear control systems on time scales. In Section 5 there is shown that two discrete-time systems defined by the forward shift operator are classically transfer equivalent if and only if their forward difference representations are transfer equivalent as systems on a time scale.

2. CALCULUS ON TIME SCALES

We give here a short introduction to differential calculus on time scales. This is a generalization of the standard differential calculus, on one hand, and the calculus of finite differences, on the other hand. Then we describe the inverse operation—integration. This will allow to solve differential equations on time scales.

More material on this subject can be found in [2].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. For $t \in \mathbb{T}$ we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

If $\sigma(t) > t$, then t is called *right-scattered*, while if $\sigma(t) < t$ is called *left-scattered*. Of $t < \sup \mathbb{T}$ and $\sigma(t) = t$ then t is called *right-dense*. If $t > \inf \mathbb{T}$ and $\sigma(t) = t$, then t is *left-dense*.
We define also the set $\mathbb{T}^k$ as: $\mathbb{T}^k := \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ if $\sup \mathbb{T} < \infty$ and $\mathbb{T}^k := \mathbb{T}$ if $\sup \mathbb{T} = \infty$. Finally, we will denote $f^\sigma := f \circ \sigma$ for any function $f : \mathbb{T} \to \mathbb{R}$.

Example 2..1

- If $\mathbb{T} = \mathbb{R}$ then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
- If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.
- Let $q > 1$. We define time scale $\mathbb{T} = q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$. Then $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$ and $\mu(t) = (q - 1)t$ for all $t \in \mathbb{T}$.

Definition 2..2 Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. *Delta derivative* of f at t, denoted by $f^\triangle(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^\triangle(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^\triangle(t)$ exists for all $t \in \mathbb{T}^k$.

In general the function σ need not be differentiable.

Remark 2..3

- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff $f^\triangle(t) = \lim\limits_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t)$, i.e. iff f is differentiable in the ordinary sense at t.
- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^\triangle(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t) = \triangle f(t)$, where $\triangle$ is the usual forward difference operator defined by the last equation above.
- If $\mathbb{T} = q^{\mathbb{Z}}$, then $f^\triangle(t) = \frac{f(qt) - f(t)}{(q-1)t}$ for all $t \in \mathbb{T} - \{0\}$.

Example 2..4 The delta derivative of t^2 is $t + \sigma(t)$. This means that the second delta derivative of t^2 may not exists.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points at $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. It can be shown that
- f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated
- σ is rd-continuous.

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D, provided $D \subset \mathbb{T}^k$, $\mathbb{T}^k \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. It can be proved that if f is regulated then there exists a function F that is pre-differentiable with region of differentiation D such that $F^\triangle(t) = f(t)$ for all $t \in D$. Any such function is called pre-antiderivative of f. Then *indefinite integral* of f is defined by $\int f(t) \triangle t := F(t) + C$ where C is an arbitrary constant. *Cauchy integral* is

$$\int_r^s f(t) \triangle t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}^k$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\triangle(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

Remark 2..5 It can be shown that every rd-continuous function has an antiderivative. Moreover, if $f(t) \ge 0$ for all $a \le t < b$ and $\int_a^b f(\tau) \triangle \tau = 0$ then $f \equiv 0$.

Example 2..6

- If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau) \triangle \tau = \int_a^b f(\tau) d\tau$, where the integral on the right is the usual Riemann integral.
- If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau) \triangle \tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

Remark 2..7 An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula of an antiderivative of t: antiderivative of $\frac{t^2}{2}$ is $\frac{t + \sigma(t)}{2} = t + \frac{\mu(t)}{2}$.

Under assumptions that: $a \in \mathbb{T}$, $\sup\mathbb{T} = \infty$ and f is rd-continuous function on $[a, \infty]$ we define *improper integral* by

$$\int_a^{\infty} f(t)\triangle\tau := \lim_{b\to\infty} \int_a^b f(t)\triangle\tau$$

provided this limit exists.

3. LAPLACE TRANSFORM

First we make some preliminary definitions. For $h > 0$. Let $Z_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im}z \leq \frac{\pi}{h}\}$ and for $h = 0$ let $Z_0 := \mathbb{C}$. Then, let us define a transformation $\xi_h : \{z \in \mathbb{C} : z \neq -\frac{1}{h}\} \rightarrow Z_h$ by $\xi_h = \frac{\mathrm{Log}(1+zh)}{h}$, where Log means the principal logarithm function. For $h = 0$ we put $\xi_0(z) := z$ for all $z \in \mathbb{C}$..

We say that function $p : \mathbb{T} \rightarrow \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. For any regressive function p let $(\ominus p)(t) := -\frac{p(t)}{1+\mu(t)p(t)}$.

Definition 3..1 If $p : \mathbb{T} \rightarrow \mathbb{R}$ is a regressive function, then the *(generalized) exponential function is defined by*

$$e_p(t, s) = \exp(\int_s^t \xi_{\mu(t)}(p(\tau))\triangle\tau) \quad \text{for all } s, t \in \mathbb{T}$$

Example 3..2 Let α be any complex constant regressive function, i.e. $\alpha \in \mathbb{C} - \{\frac{1}{h}\}$.

- If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) = e^{\alpha(t-t_0)}$.
- If $h > 0$ and $\mathbb{T} = h\mathbb{Z}$, then $e_\alpha(t, t_0) = (1 + \alpha h)^{\frac{t-t_0}{h}}$.
- If $\mathbb{T} = q^{\mathbb{Z}}$, then $e_\alpha(t, t_0) = \prod_{s\in[t_0,t)} [1 + (q - 1)\alpha s]$ for $t > t_0$.

Theorem 3..3 *Let p be regressive function and fix $t_0 \in \mathbb{T}$. Then $e_p(\cdot, t_0)$ is a unique solution of the initial value problem $x^\triangle = p(t)x$, $x(t_0) = 1$ on $\mathbb{T}$.*

Let us assume that time scale $\mathbb{T}_0$ is such that $0 \in \mathbb{T}_0$ and $\sup\mathbb{T}_0 = \infty$. From now on we will assume that $z = z(t)$ is a constant regressive function.

Definition 3..4 Assume that $x : \mathbb{T}_0 \rightarrow \mathbb{R}$ is regulated. The *Laplace transform* of x is defined as

$$\mathcal{L}\{x\}(z) := \int_0^{\infty} x(t)e^\sigma_{\ominus z}(t, 0)\triangle t$$

for $z \in D\{x\}$, where $D\{x\}$ is the set of all complex regressive constant functions for which the improper integral exists.

It can be proved that Laplace transform has the linearity property. Moreover, one can show that for all regressive constant $z \in \mathbb{C}$
- $\mathcal{L}\{1\}(z) = \frac{1}{z}$ provided $\lim_{t\to\infty} e_{\ominus z}(t, 0) = 0$ holds
- $\mathcal{L}\{x^\triangle\}(z) = zL\{x\}(z) - x(0)$ and
$\mathcal{L}\{x^{\triangle\triangle}\}(z) = z^2L\{x\}(z) - zx(0) - x^\triangle(0)$ provided $\lim_{t\to\infty} x(t)e_{\ominus z}(t, 0) = 0$ holds
- $\mathcal{L}\{e_\alpha(\cdot, 0)\}(z) = \frac{1}{z-\alpha}$ provided $\lim_{t\to\infty} e_{\alpha\ominus z}(t, 0) = 0$ holds, where $\alpha \in \mathbb{C}$ is regressive.
- If $\mathbb{T}_0$ has constant $\mu(t) \equiv h \geq 0$, then $\mathcal{L}\{x\}(z) = (1 + hz)L\{x\}(z) - h(1 + hz)x(0)$.

Example 3..5
- If $\mathbb{T}_0 = [0, \infty)$, then the Laplace transform defined above coincides with the standard $\mathcal{L}$-transformation for continuous-time case.
- If $\mathbb{T}_0 = \mathbb{N}_0$ then $(z+1)\mathcal{L}\{x\}(z) = \mathcal{Z}\{x\}(z+1)$ where $\mathcal{Z}\{x\}$ is the usual $\mathcal{Z}$-transform of x for discrete-time case.

Example 3..6 [2] Let us consider the equation

$$x^{\triangle\triangle} + 5x^\triangle + 6x = 0, \quad x(0) = 1,\ x^\triangle(0) = -5$$

defined on any time scale $\mathbb{T}_0$. Using the Laplace transform defined above we have

$$\begin{gathered} 0 = z^2L\{x\}(z) - z + 5 + \\ 5[zL\{x\}(z) - 1] + 6l\{x\}(z) \\ \mathcal{L}\{x\}(z) = \frac{3}{z+3} - \frac{2}{z+2} \end{gathered}$$

and hence

$$x(t) = 3e_{-3}(y, 0) - 2e_{-2}(t, 0)$$

for all $t \in \mathbb{T}_0$. One can notice that the continuous-time and discrete-time cases are included in this example.

Remark 3..7 In general the standard formula for the Laplace transform of the shifted function does not hold.

4. TRANSFER MATRIX. INPUT-OUTPUT AND TRANSFER EQUIVALENCE

Let Λ be a system defined by the equation

$$P(\triangle)y = Q(\triangle)u \tag{1}$$

where $y \in \mathbb{R}^r$, $u \in \mathbb{R}^m$, $\triangle$ is the differentiation operator on the time scale $\mathbb{T}$ (i.e. $\triangle f = f^\triangle$), $P(\triangle)$ and $Q(\triangle)$ are, respectively, $r \times r$ and $r \times m$ matrices whose entries are polynomials in operator $\triangle$. We assume that $\det P(\triangle) \not\equiv 0$..

Let us assume that the maximal degree of the entries of P is d_P and the maximal degree of the entries of Q is d_Q. Let $y^{(k)}(0) = 0$ for $k = 0, \ldots, d_P$ and $u^{(k)}(0) = 0$ for $k = 0, \ldots, d_Q$. Applying the Laplace transform on both sides of the input-output equation (1) we obtain

$$P(z)Y(z) = Q(z)U(z) \tag{2}$$

where $Y(z)$ and $U(z)$ are the Laplace transforms of, y and u, respectively.

As $P(z)$ is invertible for almost all $z \in \mathbb{C}$, we get $Y(z) = P(z)^{-1}Q(z)U(z)$. As usually, $G_\Lambda(z) = P(z)^{-1}Q(z)$ is called *the transfer matrix* of system Λ.

Definition 4..1 Two systems Λ_1 and Λ_2 of the form (1) are called *transfer equivalent* on time scale T if their transfer matrices are equal (as rational complex matrices).

Obviously, transfer equivalence is an equivalence relation in the set of all systems of the form (1).

Remark 4..2 If the time scale is $\mathbb{R}$, the definition of transfer equivalence is the same as in [1] (though there the Laplace transform is avoided and the transfer matrix is a rational matrix with respect to the differential operator). When $\mathbb{T} = \mathbb{Z}$, our delta operator becomes the forward difference operator and not just the forward shift as in [1]. But we show later that the two descriptions are equivalent.

Definition 4..3 Two systems Λ_1 and Λ_2 of the form (1) are called *input-output equivalent* on time scale T if they are satisfied by the same pairs (y, u).

Let us recall that a square polynomial matrix $K(z)$ is called *unimodular* if $\det K(z)$ is constant and different from 0. Then the inverse of $K(z)$ is also a polynomial unimodular matrix. It is known [11] that $K(z)$ is unimodular if and only if it can be obtained from the identity matrix by finitely many elementary row operations over the ring of polynomials in z.

Proposition 4..4 *Two input-output systems given by matrices $[P_1(z), Q_1(z)]$ and $[P_2(z), Q_2(z)]$ are transfer equivalent if and only if there are polynomials matrices $M_1(z), M_2(z)$ with $\det M_i(z) \neq 0$, $i = 1, 2$, such that*

$$M_1(z)[P_1(z), Q_1(z)] = M_2(z)[P_2(z), Q_2(z)] \tag{3}$$

Proof: The proof is standard. Let us assume that systems Λ_1 and Λ_2 are transfer equivalent, i.e. $G_{\Lambda_1}(z) = P_1^{-1}(z)Q_1(z) = P_2^{-1}(z)Q_2(z) = G_{\Lambda_2}(z)$. Let $d = \det P_1(z) \det P_2(z)$ and define $M_1(z) := dP_1^{-1}(z)$, $M_2(z) = dP_2^{-1}(z)$. Then $\det M_i(z) \neq 0$, $i = 1, 2$, and

$$dM_1^{-1}(z)[P_1(z), Q_1(z)] = [dI, dP_1^{-1}(z)Q_1(z)]$$
$$dM_2^{-1}(z)[P_2(z), Q_2(z)] = [dI, dP_2^{-1}(z)Q_2(z)]$$

So we get (3).

Now let us assume that there exist polynomial matrices $M_1(z), M_2(z)$ such that (3) holds. Then

$$\begin{aligned} G_{\Lambda_1}(z) = P_1^{-1}(z)Q_1(z) = \\ M_1(z)P_1(z))^{-1}M_1(z)Q_1(z) = \\ (M_2(z)P_2(z))^{-1}M_2(z)Q_2(z) = \\ P_2^{-1}(z)Q_2(z) = G_{\Lambda_2}(z). \quad \Box \end{aligned}$$

Proposition 4..5 *Two input-output systems given by matrices $[P_1(z), Q_1(z)]$ and $[P_2(z), Q_2(z)]$ are input-output equivalent if and only if there is a unimodular matrix $K(z)$ such that $[P_1(z), Q_1(z)] = K(z)[P_2(z), Q_2(z)]$.*

Proof: See [1] for the continuous-time case. The general case of arbitrary time scale is similar. □

Corollary 4..6 *If two systems Λ_i, $i = 1, 2$ are input-output equivalent then they are transfer equivalent.*

5. TRANSFER EQUIVALENCE FOR DISCRETE-TIME SYSTEMS

We shall study here two descriptions of discrete-time systems. The first one is given by the forward time-shift operator, the second one by the forward time difference operator.

Let $\mathbb{T} = \mathbb{Z}$. Then $\triangle$ acts now on a sequence $(f(k))$ by

$$(\triangle f)(k) = f(k+1) - f(k), \tag{4}$$

so it is now the forward time difference operator. Let δ denote the forward time-shift operator

$$(\delta f)(k) = f(k+1). \tag{5}$$

If id denotes the identity map, then $\triangle = \delta - \mathrm{id}$ and conversely $\delta = \triangle + \mathrm{id}$.

Usually one studies discrete-time linear systems defined with the aid of the time-shift operator, either in the state-space form

$$\delta x = Ax + Bu \tag{6}$$
$$y = Cx + Du \tag{7}$$

or in the more general form

$$\Gamma : \quad M(\delta)y = N(\delta)u, \tag{8}$$

where M and N are polynomial matrices.

Assuming that sufficiently many elements of the sequences $(y(0), y(1), \ldots)$ and $(u(0), u(1), \ldots)$ are zero and applying the standard $\mathcal{Z}$-transformation to (8) we can compute the standard transfer matrix of (8)

$$H_\Gamma(z) = M(z)^{-1}N(z). \tag{9}$$

We say that two systems Γ_1 and Γ_2 of the form (8) are *classically transfer equivalent* if their standard transfer matrices coincide.

Now replacing δ by $\triangle + \text{id}$ we can transform the system Γ to a differential system on the time scale $\mathbb{Z}$:

$$\Lambda : \ P(\triangle)y = Q(\triangle)u, \tag{10}$$

where P and Q are some polynomial matrices.

Observe that transformation of Γ to Λ is invertible. Replacing $\triangle$ by $\delta - \text{id}$ in the equation (10) defining Λ we can recover the system Γ. Let T denote the map that assigns to a system Γ the system Λ, so we can write $\Lambda = T(\Gamma)$.

As Λ is a differential system on a time scale, we can define its transfer matrix and study transfer equivalence of two systems of this form. We can prove now the following:

Proposition 5..1 *Two discrete-time systems Γ_1 and Γ_2 are classically transfer equivalent if and only if the systems $T(\Gamma_1)$ and $T(\Gamma_1)$ are transfer equivalent.*

Proof: Let $\Lambda_i = T(\Gamma_i)$: $P_i(\triangle)y = Q_i(\triangle)u$ for $i = 1, 2$. Then $P_i(\triangle) = M_i(\triangle + \text{id})$ and $Q_i(\triangle) = N_i(\triangle + \text{id})$ for $i = 1, 2$. Observe that

$$\mathcal{L}\{(\triangle + \text{id})^n\} = (z + 1)^n$$

for $n \in \mathbb{N}$ (we treat $(\triangle + \text{id})^n$ as an operator on functions). Thus $\mathcal{L}\{P_i(\triangle)\} = M_i(z + 1)$, $\mathcal{L}\{Q_i(\triangle)\} = N_i(z + 1)$, $i = 1, 2$, and

$$\begin{aligned} G_{\Lambda_i}(z) &= P_i(z)^{-1}Q_i(z) \\ &= M_i(z+1)^{-1}N_i(z+1) = H_{\Gamma_i}(z+1). \end{aligned}$$

The systems Γ_1 and Γ_2 are classically transfer equivalent if and only if the standard transfer matrices H_{Γ_1} and H_{Γ_2} are equal. This holds if and only if the transfer matrices of Λ_1 and Λ_2 are equal which means that Λ_1 and Λ_2 are equivalent. □.

Similar result for SISO systems was proved in [4].

6. CONCLUDING REMARKS

We have generalized the notation of transfer matrix and defined the transfer equivalence for linear control systems described by input-output polynomial equations (in delta derivative operator). Moreover, we have given a characterization of this property and shown what it means for the time scale of integer numbers. A further problem to be studied is the irreducibility of the input-output equations and its reduction if equations are reducible, and to extend the time-scales approach to the nonlinear case.

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