External Dynamical Equivalence of Analytic Control Systems on Homogeneous Time Scales

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Abstract.
Theory of systems on homogeneous time scales unifies theories of continuous-time and discrete-time systems. The characterizations of external dynamic equivalence known for continuous-time and discrete-time systems are extended here to systems on time scales. Under assumption of uniform observability, it is shown that two analytic control systems with output are externally dynamically equivalent if and only if their delta universes are isomorphic. The delta operator associated to the system on a time scale is a generalization of the differential operator associated to a continuous-time system and of the difference operator associated to a discrete-time system.

1. Introduction. In 1988, in his Ph.D. thesis [13], Stefan Hilger developed calculus on time scales, which unified the standard differential calculus and the calculus of finite differences. This allowed for a unified treatment of dynamical systems with continuous and discrete time. The book by M. Bohner and A. Peterson [7] contains the most important achievements in this area. But the theory is much richer than just the unification. One can study systems for which time is partly continuous and partly discrete. Many systems appearing in engineering, biology and economy exhibit such features.

In the classical control theory there always have been two parallel areas of research: continuous-time and discrete-time systems. Most of the results are similar for both classes of systems, but there are also significant differences. For example, asymptotic stability of linear time-invariant systems is characterized by the condition that the eigenvalues of the system lie in the specific region of the complex plane. However this region depends on the class of systems. Another difference concerns solutions of differential equations (continuous time) and difference equations (discrete time). Under some reasonable conditions ordinary differential equations can be solved forward and backward, while difference equations usually can be solved only forward and some extra assumptions are required to solve them backward. Fortunately, in control theory we are mostly interested in forward solutions, so this difference is less important.

Calculus on time scales entered control theory just a few years ago. First results concerned basic properties of linear systems, like controllability, observability and realizations (see [4, 9, 6]). In [5] we studied dynamic feedback equivalence of nonlinear systems on time scales. The main result of that paper will be used here to show a characterization of external dynamic equivalence for systems on time scales. Another attempt to unify continuous-time and discrete-time systems, without use of calculus on time scales, was made in [11, 18].

Dynamic equivalence for nonlinear continuous-time systems was first studied by B. Jakubczyk [14, 15]. He used a dynamic state feedback to transfer trajectories of one system onto trajectories of the other. His concept of dynamic feedback linearizability was close to the property of flatness introduced earlier by M. Fliess (see e.g. [10, 17] and [22]). The main result of [14] says that two systems are dynamically (state) feedback equivalent if and only if their differential algebras are isomorphic. In [2] dynamic state feedback equivalence of discrete-time systems was studied. In the characterization of this property, difference algebras were used instead of differential algebras. This was one of the examples where the

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results for continuous time and discrete time are close to each other and a clever change of language is enough to switch between two classes of systems.

In [20, 3] external dynamic equivalence and linearization for discrete-time systems were studied. The systems were equipped with output parts and dynamic output feedback was used instead of dynamic state feedback. Necessary and sufficient criteria of external dynamic equivalence were expressed with the aid of the output difference universe. The concept of (function) universe, introduced by J. Johnson, is a generalization of the notion of (function) algebra [1, 16]. Besides standard algebraic operations possible in algebras, the structure of universe allows for substituting elements of the universe (which are partially defined functions) into real-analytic functions of several variables and amalgamation of partially defined functions from the universe. The results of [20, 3] were then transferred (back) to continuous-time systems, with the output differential universe as the key tool [21].

In [5] we studied dynamic state feedback equivalence for nonlinear systems on homogeneous time scales. The results unified those obtained for continuous-time and discrete-time systems. Instead of differential and difference operators used in earlier works, we introduced so called delta operator, which for a particular time scale would become either a differential operator or a difference one.

In this paper we complete the picture studying external dynamic equivalence for control systems with outputs, defined on homogeneous time scales. The main result says that two systems are externally dynamically equivalent if and only if their delta universes are isomorphic. This theorem may be seen as an extension of the result of [5] to systems with output or as a unification of our earlier results from [20] and [21]. We assume that the systems are uniformly observable and use the main theorem of [5] to prove the present version. In [3] and [21] we showed that for continuous-time and discrete-time cases one can drop the observability assumptions. Only some regularity of the space obtained by gluing up indistinguishable states of the system is assumed. This suggests that the same could be done for systems on time scales.

The paper is organized as follows. In Section 2 we provide the reader with the necessary background on the calculus on time scales. Section 3 contains setting of the problem and the precise definition of external dynamic equivalence. In Section 4 we recall basic concepts of the theory of universes. The main result of the paper is stated and proved in Section 5.

2. Calculus on time scales.
We recall here basic concepts and facts of the calculus on time scales. For more information the reader is referred to [7]

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of real numbers $\mathbb{R}$. The standard examples of time scales are $\mathbb{R}$, $h\mathbb{Z}$, $h > 0$, $\mathbb{N}$, $\mathbb{N}_0$, $2^{\mathbb{N}_0}$ or $\mathbb{P}_{a,b} = \bigcup_{k=0}^{\infty} [k(a + b), k(a + b) + a]$. The time scales $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$.

The following operators on $\mathbb{T}$ are often used:

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$, if $\sup \mathbb{T} \in \mathbb{T}$,

- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$, defined by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$, if $\inf \mathbb{T} \in \mathbb{T}$,

- the *graininess functions* $\mu, \nu : \mathbb{T} \to [0, \infty[$ defined by $\mu(t) := \sigma(t) - t$ and respectively by $\nu(t) := t - \rho(t)$.

Points from the time scale can be classified as follows: a point $t \in \mathbb{T}$ is called

- *right-scattered* if $\sigma(t) > t$ and *right-dense* if $\sigma(t) = t$,

- *left-scattered* if $\rho(t) < t$ and *left-dense* if $\rho(t) = t$,

- *isolated* if it is both left-scattered and right-scattered,

- *dense* if it is both left-dense and right-dense.

We define also the sets

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty. \end{cases}, \quad \mathbb{T}_\kappa := \begin{cases} \mathbb{T} \setminus \{m\}, & \text{if } m \text{ is right scattered minimum} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

Example 2.1. *If* $\mathbb{T} = \mathbb{R}$ *then* $\rho(t) = t = \sigma(t)$ *and* $\mu(t) = 0$, *for all* $t \in \mathbb{R}$.

Example 2.2. *If* $\mathbb{T} = h\mathbb{Z}$, $h > 0$, *then* $\rho(t) = t - h$, $\sigma(t) = t + h$ *and* $\mu(t) \equiv h$, *for all* $t \in h\mathbb{Z}$.

Example 2.3. *If* $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \left\{ q^k : \; k \in \mathbb{Z} \right\} \cup \{0\}$, *where* $q > 1$, *then* $\rho(t) = \frac{t}{q}$ $\sigma(t) = qt$ *and* $\mu(t) = (q-1)t$, *for all* $t \in \mathbb{T}$.

Definition 2.4. *A time scale* $\mathbb{T}$ *is called* homogeneous *if* μ *is constant on* $\mathbb{T}^\kappa$.

The time scales $\mathbb{R}$ and $h\mathbb{Z}$ are homogeneous, whereas $\overline{q^{\mathbb{Z}}}$ is not. In this paper we will be interested in homogeneous time scales. Thus we rather want to unify the continuous-time and discrete-time cases, and not to develop a general theory.

Definition 2.5. *Let* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}^k$. *Then the number* $f^\Delta(t)$ *(when it exists), with the property that, for any* $\varepsilon > 0$, *there exists a neighborhood* U *of* t *such that*

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \le \varepsilon |\sigma(t) - s|, \; \forall s \in U,$$

is called the delta derivative *of* f *at* t.
    The function $f^\Delta : \mathbb{T}^k \to \mathbb{R}$ *is called the* delta derivative *of* f *on* $\mathbb{T}^\kappa$.
    We say that f *is* delta differentiable *on* $\mathbb{T}^\kappa$, *if* $f^\Delta(t)$ *exists for all* $t \in \mathbb{T}^\kappa$.

Definition 2.6. *Let* $f : \mathbb{T} \to \mathbb{R}$ *and* $t \in \mathbb{T}_k$. *Then the number* $f^\nabla(t)$ *(when it exists), with the property that, for any* $\varepsilon > 0$, *there exists a neighborhood* U *of* t *such that*

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)[\rho(t) - s]| \le \varepsilon |\rho(t) - s|, \; \forall s \in U,$$

is called the nabla derivative *of* f *at* t.
    The function $f^\nabla : \mathbb{T}_k \to \mathbb{R}$ *is called the* nabla derivative *of* f *on* $\mathbb{T}_\kappa$.
    We say that f *is* nabla differentiable *on* $\mathbb{T}_\kappa$, *if* $f^\nabla(t)$ *exists for all* $t \in \mathbb{T}_\kappa$.

Remark 2.7. *If* $\mathbb{T} = \mathbb{R}$, *then* $f : \mathbb{R} \to \mathbb{R}$ *is both delta differentiable and nabla differentiable at* $t \in \mathbb{R}$ *iff*

$$f^\Delta(t) = f^\nabla(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t),$$

i.e. f *is differentiable in the ordinary sense at* t.
If $\mathbb{T} = \mathbb{Z}$, *then* $f : \mathbb{Z} \to \mathbb{R}$ *is always delta differentiable and nabla differentiable on* $\mathbb{Z}$ *and*

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} = f(t+1) - f(t),$$

$$f^\nabla(t) = \frac{f(t)) - f(\rho(t))}{\mu(t)} = f(t) - f(t-1),$$

for all $t \in \mathbb{Z}$.

Proposition 2.8. *[12]*

    i) Assume that $f : \mathbb{T} \to \mathbb{R}$ *is delta differentiable on* $\mathbb{T}^k$. *Then* f *is nabla differentiable at* t *and*

$$f^\nabla(t) = f^\Delta(\rho(t)) \tag{1}$$

    for $t \in \mathbb{T}_k$ *such that* $\sigma(\rho(t)) = t$. *If, in addition,* f^Δ *is continuous on* $\mathbb{T}^k$, *then* f *is nabla differentiable at* t *and (1) holds for any* $t \in \mathbb{T}_k$.

ii) Assume that $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable on $\mathbb{T}_k$. Then f is delta differentiable at t and

$$f^{\Delta}(t) = f^{\nabla}(\sigma(t)) \tag{2}$$

for $t \in \mathbb{T}^k$ such that $\rho(\sigma(t)) = t$. If, in addition, f^{∇} is continuous on $\mathbb{T}_k$, then f is delta differentiable at t and (2) holds for any $t \in \mathbb{T}^k$.

Remark 2.9. *If $t \in \mathbb{T}^{\kappa}$ satisfies $\rho(t) = t < \sigma(t)$, then the forward jump operator σ is not delta differentiable at t. Clearly, the forward jump operator σ is not differentiable at points that are left-dense and right-scattered at the same time (then σ is not continuous).*

Remark 2.10. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* regulated *if its right-side limits exist (finite) at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at all left-dense points in $\mathbb{T}$.*

Definition 2.11. *A function $f : \mathbb{T} \to \mathbb{R}$ is called* rd-continuous *if it is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.*

The set of all rd-continuous functions is denoted by $\mathcal{C}_{rd}$. It may be shown [7] (Theorem 1.60) that

- f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated

- σ is rd-continuous.

Definition 2.12. *A continuous function $f : \mathbb{T} \to \mathbb{R}$ is* pre-differentiable *with D (the region of differentiation), if $D \subset \mathbb{T}^{\kappa}$, $\mathbb{T}^{\kappa} \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$ and f is differentiable at each $t \in D$.*

If f is regulated then there is a function F that is pre-differentiable with the region of differentiation D such that $F^{\Delta}(t) = f(t)$, for all $t \in D$. Any function F that satisfies $F^{\Delta}(t) = f(t)$ is called a *pre-antiderivative* of f. Then the *indefinite integral* of a regulated function f is defined by $\int f(t)\Delta t = F(t) + C$, where C is arbitrary constant. The *Cauchy integral* of a regulated function f is defined by $\int_{r}^{s} f(t)\Delta t = F(s) - F(r)$, for all $s, t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *antiderivative of $f : \mathbb{T} \to \mathbb{R}$* if it satisfies $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^{\kappa}$.

It is known that every rd-continuous function has an antiderivative.

Example 2.13. *If $\mathbb{T} = \mathbb{R}$, then*

$$\int_{a}^{b} f(t)\Delta t = \int_{a}^{b} f(t)dt,$$

where the integral on the right is the usual Riemann integral.

Example 2.14. *If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then*

$$\int_{a}^{b} f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h \cdot f(kh),$$

for $a < b$.

3. External dynamic equivalence. Let $\mathbb{T}$ be a homogeneous time scale. Consider a system Σ of the following form:

$$
\begin{aligned}
x^\Delta(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t))
\end{aligned}
\tag{3}
$$

where $t \in \mathbb{T}$, $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$. We shall assume that f and h are analytic, and that control u may be infinitely many times differentiated as a function on the time scale $\mathbb{T}$.

By a *trajectory* of system Σ we will mean any triple of functions $(y(\cdot), x(\cdot), u(\cdot))$ defined on some interval that satisfies the equations of Σ. Then the pair $(x(\cdot), u(\cdot))$ is an *inner trajectory* and the pair $(y(\cdot), u(\cdot))$ is an *external trajectory* of Σ. The set of all inner (external) trajectories of the system Σ forms the *inner (external) behavior* of the system. It will be denoted by $B^i(\Sigma)$ (respectively by $B^e(\Sigma)$).

By $J(r)$ we will denote the space of all infinite sequences $S = (s^{(i)})_{i \geq 0}$, $s^{(i)} \in \mathbb{R}^r$. If $z : \mathbb{T} \to \mathbb{R}^r$ has infinitely many delta-derivatives, the map $Z : \mathbb{T} \to J(r)$ defined by $Z(t) = (z(t), z^\Delta(t), \ldots)$ is called the (infinite) jet of z. We shall consider real maps defined on $J(r)$. We assume that each such map φ depends only on a finite number of elements of the sequence S (but the number of these elements depends on the given map φ). In this case we say that the function φ is *finitely presented*. A *map* $\phi : J(r) \to \mathbb{R}^{\tilde{r}}$ is *finitely presented* if all its components have this property.

Let us consider two systems on the time scale $\mathbb{T}$:

$$
\Sigma : \quad
\begin{aligned}
x^\Delta(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t))
\end{aligned}
\quad \text{and} \quad
\tilde{\Sigma} : \quad
\begin{aligned}
\tilde{x}^\Delta(t) &= \tilde{f}(\tilde{x}(t), \tilde{u}(t)) \\
\tilde{y}(t) &= \tilde{h}(\tilde{x}(t))
\end{aligned}
$$

where $x(t) \in \mathbb{R}^n$, $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $y(t) \in \mathbb{R}^p$, $\tilde{y}(t) \in \mathbb{R}^{\tilde{p}}$, $u(t), \tilde{u}(t) \in \mathbb{R}^m$, $t \in \mathbb{T}$.

Consider the following dynamic transformations:

$$
y = \phi^e(\tilde{Y}, \tilde{U}), \quad u = \psi^e(\tilde{Y}, \tilde{U})
\tag{4}
$$

$$
\tilde{y} = \tilde{\phi}^e(Y, U), \quad \tilde{u} = \tilde{\psi}^e(Y, U)
\tag{5}
$$

where $Y \in J(p)$, $\tilde{Y} \in J(\tilde{p})$, $U, \tilde{U} \in J(m)$, and $\phi^e, \tilde{\phi}^e, \psi^e, \tilde{\psi}^e$ are finitely presented maps of the class C^ω.

We say that Σ and $\tilde{\Sigma}$ are *externally dynamically equivalent*, if there exist transformations (4) and (5) that induce the following relations between the external trajectories of both systems:

$$
\begin{aligned}
y(t) &= \phi^e(\tilde{y}(t), \tilde{y}^\Delta(t), \ldots, \tilde{y}^{\Delta^k}(t), \tilde{u}(t), \tilde{u}^\Delta(t), \ldots, \tilde{u}^{\Delta^k}(t)) \\
u(t) &= \psi^e(\tilde{y}(t), \tilde{y}^\Delta(t), \ldots, \tilde{y}^{\Delta^k}(t), \tilde{u}(t), \tilde{u}^\Delta(t), \ldots, \tilde{u}^{\Delta^k}(t))
\end{aligned}
\tag{6}
$$

and similarly for $(\tilde{y}(t), \tilde{u}(t))$.

Remark 3.1. *Proposition 2.8. implies that external dynamic equivalence of two nonlinear systems can be replaced by so called* external anticipating equivalence *of these systems. In general, we say that Σ and $\tilde{\Sigma}$ are* external anticipating equivalent, *if there exist transformations* (4) *and* (5) *that induce the following relations between the external trajectories of both systems:*

$$
\begin{aligned}
y(t) &= \phi^e(\tilde{y}(t), \tilde{y}^\nabla(t), \ldots, \tilde{y}^{\nabla^k}(t), \tilde{u}(t), \tilde{u}^\nabla(t), \ldots, \tilde{u}^{\nabla^k}(t)) \\
u(t) &= \psi^e(\tilde{y}(t), \tilde{y}^\nabla(t), \ldots, \tilde{y}^{\nabla^k}(t), \tilde{u}(t), \tilde{u}^\nabla(t), \ldots, \tilde{u}^{\nabla^k}(t))
\end{aligned}
\tag{7}
$$

and similarly for $(\tilde{y}(t), \tilde{u}(t))$.

External dynamical equivalence is an equivalence relation in the set of all control systems with output. It is a natural generalization of similar concepts for continuous-time and discrete-time systems.

We say that the system Σ is *externally dynamically linearizable* if it is externally dynamically equivalent to a linear minimal one (i.e. controllable and observable).

To state a characterization of external dynamical equivalence, we need the concept of function universe.

4. Function universes. Let X, Y be sets. A *partially defined function* on X with values in Y is any map $\varphi : A \rightarrow Y$, where $A \subseteq X$ is called domain of φ and denoted by $\text{dom}\varphi$. If $\text{dom}\varphi = \text{X}$ then φ is *global*. Let Y_X be the set of all partially defined functions on X. One can extend any $\varphi \in Y_X$ to one defined on X by assigning $\varphi(x) = \emptyset_0$ for $x \notin \text{dom}\varphi$. We call $\emptyset_0$ the *phantom*. Now $\text{dom}\varphi = \{\text{x} \in \text{X} : \varphi(\text{x}) \neq \emptyset_0\}$. If $a \in Y$, $x \in X$ then we set $a_X(x) := a$.

Let A_n denote the set of functions of class C^ω, partially defined on open subsets in $\mathbb{R}^n$ with values in $\mathbb{R}$. In particular, A_0 can be identified with $\mathbb{R} \cup \emptyset_0$. The topology in A_0 can be defined as follows: a subset $B \subset A_0$ is open if $B = A_0$ or B is an open subset in $\mathbb{R}$.

Functions $\varphi, \psi \in Y_X$ are *matching*, if they take on the same values on $\text{dom}\varphi \cap \text{dom}\psi$. Let us consider a set $M \subseteq Y_X$ of functions that are matching and define a function $\underline{M} \in Y_X$: $\underline{M}(x) = \emptyset_0$ if no function in M is defined at x and $\underline{M}(x) = \varphi(x)$ for any function $\varphi \in M$ defined at x. The process of constructing $\underline{M}$ is called *amalgamation* of the functions of M.

Let $\varphi_1, \ldots, \varphi_k \in \mathbb{R}_X$ and $F \in A_k$. Then $F \circ (\varphi_1, \ldots, \varphi_k)$ is a partially defined function on X given by

$$(F \circ (\varphi_1, \ldots, \varphi_k))(x) = F(\varphi_1(x), \ldots, \varphi_k(x))$$

for $x \in X$. If $\varphi_i(x) = \emptyset_0$ or $(\varphi_1(x), \ldots, \varphi_k(x)) \notin \text{dom}F$ then $F(\varphi_0(x), \ldots, \varphi_n(x)) = \emptyset_0$. The map

$$(\varphi_1, \ldots, \varphi_k) \mapsto F \circ (\varphi_1, \ldots, \varphi_k)$$

is called a *substitution*.

A set $\mathcal{U} \subseteq \mathbb{R}_X$ containing 0_X and closed under substitutions and amalgamation is called a *function universe on the set X* [16]. A *function subuniverse* of the universe $\mathcal{U}$ is a subset $\hat{U} \subset \mathcal{U}$ that is a function universe on X. If $\mathcal{H} \subset \mathcal{U}$, then function *subuniverse generated by $\mathcal{H}$* is the smallest subuniverse of $\mathcal{U}$ containing $\mathcal{H}$ [1].

In a natural way a function universe $\mathcal{U}$ on X induces a topology on X: the open sets have the form $\text{dom}\varphi : \varphi \in \mathcal{U}$.

Let $\mathcal{U}_1, \mathcal{U}_2$ be function universes on X_1 and X_2 respectively. A map $\tau : \mathcal{U}_1 \rightarrow \mathcal{U}_2$ is a *homomorphism* of function universes $\mathcal{U}_1$ and $\mathcal{U}_2$ if
1. $\tau(F \circ (\varphi_1, \ldots, \varphi_k)) = F(\tau\varphi_1, \ldots, \tau\varphi_k)$ for $\varphi_1, \ldots, \varphi_k \in \mathcal{U}_1$, $F \in A_k$
2. $\tau(\underline{M}) = \overline{\tau(M)}$ for any matching set $M \subset \mathcal{U}_1$
3. $\tau(0_{X_1}) = \overline{0_{X_2}}$
If a homomorphism τ is a bijective map then it is an *isomorphism* of function universes.

5. Conditions of equivalence. We shall assume the following conditions on the dynamics of the system (3):

Condition D1. For every $x, y \in \mathbb{R}^n$ there is at most one u that satisfies the equation

$$y = f(x, u). \tag{8}$$

Condition D2. For any x and u the rank of the matrix

$$\frac{\partial f}{\partial u}(x, u)$$

is full (i.e. equal m).

Condition D3. The map

$$\mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \times \mathbb{R}^n : (x, u) \mapsto (x, f(x, u))$$

is proper, i.e. the inverse image of a compact set in $\mathbb{R}^n \times \mathbb{R}^n$ is a compact set in $\mathbb{R}^n \times \mathbb{R}^m$.

Conditions D1-D3 were first introduced in [14, 15] to prove criteria of dynamic feedback equivalence for nonlinear continuous-time systems. Later there were used in [2] to show a similar result for nonlinear discrete-time systems, and recently, under the same assumptions, we have unified these characterizations stating the dynamic feedback equivalence criteria for nonlinear systems on homogeneous time scales [5].

The dynamic feedback equivalence concerns only the dynamical part of the system Σ, i.e. the equation

$$x^\Delta(t) = f(x(t), u(t)).$$

Two systems are said to be *dynamically feedback equivalent* if they are externally dynamically equivalent after replacing their output functions by the identity maps (i.e. setting $y(t) = x(t)$). Then the dynamic transformations maps ϕ^e and ψ^e, denoted now by ϕ and ψ, depend rather on delta-derivatives of x and u and not of y and u.

Let T be the map $\mathbb{R}^n \times J(m) \to J(n)$ defined by

$$T(x_0, U) := X = (x^{(0)}, x^{(1)}, \ldots),$$

where (X, U) is the infinite jet at $t = 0$ of the inner trajectory (x, u) of Σ that satisfies the initial condition $x(0) = x_0$. One can show that such X is unique which means that T is well defined. Moreover, Conditions D1 and D2 imply that T is left invertible. Thus, if X is the jet of the solution x corresponding to a control u (and the initial condition $x(0) = x_0 = x^{(0)}(0)$), then the jet of u can be expressed as $U = S(X)$, where components of S are finitely presented and analytic. This means that the dynamic feedback equivalence is given by the equations

$$x(t) = \phi(\tilde{x}(t), \tilde{x}^\Delta(t), \ldots, \tilde{x}^{\Delta^k}(t))$$
$$u(t) = \psi(\tilde{x}(t), \tilde{x}^\Delta(t), \ldots, \tilde{x}^{\Delta^k}(t), \tilde{u}(t), \tilde{u}^\Delta(t), \ldots, \tilde{u}^{\Delta^k}(t)) \tag{9}$$

and similarly for $(\tilde{x}(t), \tilde{u}(t))$.

Let $A(n, m)$ denote the algebra of all finitely presented analytic functions $\varphi : \mathbb{R}^n \times J(m) \to \mathbb{R}$ and let Σ be given by (3). Let us define the operator $\delta_\Sigma : A(n, m) \to A(n, m)$, associated with Σ, by

$$(\delta_\Sigma \varphi)(x, U) :=$$
$$\int_0^1 \frac{\partial \varphi}{\partial x}(x + h\mu f(x, u^{(0)}), U)dh \cdot f(x, u^{(0)}) +$$
$$\sum_{i=0}^\infty \int_0^1 \frac{\partial \varphi}{\partial u^{(i)}}(x, U + h\mu U_1))dh \cdot u^{(i+1)} \tag{10}$$

where $U_1 = (u^{(1)}, u^{(2)}, \ldots)$. It is clear that this operator depends only on the dynamical part of Σ. It will be called the delta operator of the system Σ.

Remark 5.1. *The delta operator has the following interpretation. Let $U(\cdot) = (u(\cdot), u^\Delta(\cdot), \ldots)$ be the infinite jet of control u and let $x(\cdot)$ be the solution of (3) corresponding to u and the initial condition $x(0) = x_0$. Then the delta derivative at $t = 0$ of $t \to \varphi(x(t), U(t))$ is equal to $(\delta_\Sigma \varphi)(x_0, U(0))$. If $\mathbb{T} = \mathbb{R}$, then δ_Σ is a derivation of the algebra $A(n, m)$, i.e. it is linear and satisfies the Leibniz rule. This is not the case for other time scales. But because of the above interpretation we shall often call $\delta_\Sigma \varphi$ the δ_Σ derivative of φ.*

The algebra $A(n, m)$ together with the operator δ_Σ is called the *delta algebra of the system Σ* and denoted by A_Σ. A homomorphism of delta algebras A_Σ and $A_{\tilde{\Sigma}}$ is a homomorphism $\tau : A(n, m) \to A(\tilde{n}, m)$ of algebras that satisfies the condition $\delta_{\tilde{\Sigma}} \circ \tau = \tau \circ \delta_\Sigma$. An *isomorphism* of the delta algebras A_Σ and $A_{\tilde{\Sigma}}$ is a homomorphism that is a bijective map.

The main result of [5] says the following

Theorem 5.2. *Systems Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent iff their delta algebras A_Σ and $A_{\tilde{\Sigma}}$ are isomorphic.*

Now we are going to show a similar characterization of external dynamical equivalence for systems on homogeneous time scales.

Let $\mathcal{A}(n, m)$ denote the function universe of all partially defined and finitely presented analytic functions on $\mathbb{R}^n \times J(m)$. It is generated by the algebra $A(n, m)$. Observe that the operator δ_Σ can naturally be extended to $\mathcal{A}(n, m)$. For a system Σ given by (3) we define the *observation universe* $\mathcal{U}_\Sigma$ to be the smallest subuniverse of $\mathcal{A}(n, m)$ containing the components h_i, $i = 1, \ldots, r$, of the map h, the coordinate functions u_j, $j = 1, \ldots, m$, and invariant under the action of δ_Σ.

The observation universe $\mathcal{U}_\Sigma$ together with the operator δ_Σ is called the *delta universe of the system* Σ (and will be denoted by the same symbol $\mathcal{U}_\Sigma$). A morphism of delta universes $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$ is a morphism $\tau : \mathcal{U}_\Sigma \to \mathcal{U}_{\tilde{\Sigma}}$ of function universes that satisfies the condition

$$\delta_{\tilde{\Sigma}} \circ \tau = \tau \circ \delta_\Sigma.$$

An *isomorphism* of the delta universes $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$ is a morphism that is a bijective map.

The system Σ is called *uniformly observable* if every coordinate function x_i, $i = 1, \ldots, n$, belongs to $\mathcal{U}_\Sigma$.

Remark 5.3. *Uniform observability means that locally one can express coordinate functions as analytic functions of the output function h, the control u and their δ_Σ 'derivatives'. This means that for every control function u, any two distinct initial states can be distinguished by observing the output. Moreover, we can recover the state from the output, the control and their derivatives. As in [20] we could assume specific conditions on h and f that guarantee uniform observability of the system. However we want to concentrate here on the problem of dynamic equivalence, so we put aside all the details concerning observability.*

Proposition 5.4. *The system Σ is uniformly observable iff $\mathcal{U}_\Sigma = \mathcal{A}(n, m)$.*

Proof. If Σ is uniformly observable, then for every $i = 1, \ldots, n$, x_i belongs to $\mathcal{U}_\Sigma$. By definition, $\mathcal{U}_\Sigma$ contains also u_j for $j = 1, \ldots, m$ and all $\delta_\Sigma^k u_j = u_j^{(k)}$. Substituting x_i, $i = 1, \ldots, n$, and $u_j^{(k)}$, $j = 1, \ldots, m$, $k \geq 0$, into analytic partially defined functions we get all the elements of $\mathcal{A}(n, m)$. On the other hand, if $\mathcal{U}_\Sigma = \mathcal{A}(n, m)$, then every x_i, $i = 1, \ldots, n$, belongs to $\mathcal{U}_\Sigma$. ∎

Corollary 5.5. *Assume that two systems Σ and $\tilde{\Sigma}$ are uniformly observable. Then, A_Σ and $A_{\tilde{\Sigma}}$ are isomorphic iff $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$ are isomorphic.*

Proof. From uniform observability we get $\mathcal{U}_\Sigma = \mathcal{A}(n, m)$ and $\mathcal{U}_{\tilde{\Sigma}} = \mathcal{A}(\tilde{n}, m)$. As $A(n, m)$ and $A(\tilde{n}, m)$ are the algebras of global functions of the universes $\mathcal{A}(n, m)$ and $\mathcal{A}(\tilde{n}, m)$, respectively, then the restriction of an isomorphism of $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$ gives an isomorphism of A_Σ and $A_{\tilde{\Sigma}}$. The other implication follows from the specific form of any isomorphism τ of A_Σ and $A_{\tilde{\Sigma}}$ (see [5]). It is a pullback of a certain map, so it commutes with substitutions and amalgamations. Thus the isomorphism τ of A_Σ and $A_{\tilde{\Sigma}}$ is also an isomorphism of the function universes $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$. This also allows to show that the condition $\delta_{\tilde{\Sigma}} \circ \tau = \tau \circ \delta_\Sigma$ can be extended to the universes $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$. ∎

If θ is a real function on $\mathbb{R}^n \times J(m)$, then by Θ we denote its infinite *delta-jet*: $(\theta, \delta_\Sigma(\theta), \delta_\Sigma^2(\theta), \ldots)$. This, in particular, concerns the coordinate functions x_i and u_j and their aggregations x and u, whose delta-jets are denoted by X and U.

Uniform observability of Σ means that $x = \varrho(Y, U)$ for some finitely presented analytic map ϱ. Thus $X = R(Y, U)$, where R is the infinite delta-jet of ϱ. This equation may be rewritten on the level of jets of functions of time: $X(t) = R(Y(t), U(t))$. On the other hand, from the relations $y = h(x)$, after applying the operator δ_Σ, we get $Y = \Gamma(X, U)$, for some map Γ with finitely presented components.

Proposition 5.6. *Assume that two systems Σ and $\tilde{\Sigma}$ are uniformly observable. Then, Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent iff they are externally dynamically equivalent.*

Proof. Assume first that Σ and $\tilde{\Sigma}$ are dynamically feedback equivalent. Thus there exist transformations ϕ, ψ, $\tilde{\phi}$ and $\tilde{\psi}$ such that for an inner trajectory $(\tilde{x}, \tilde{u})$ of $\tilde{\Sigma}$ the equations

$$x(t) = \phi(\tilde{X}(t))$$
$$u(t) = \psi(\tilde{X}(t), \tilde{U}(t))$$

define an inner trajectory of Σ, and similarly for $(\tilde{x}(t), \tilde{u}(t))$. Thus

$$y(t) = h(\phi(\tilde{X}(t)))$$
$$u(t) = \psi(\tilde{X}(t), \tilde{U}(t))$$

define an external trajectory of Σ. From uniform observability, $\tilde{X} = R(\tilde{Y}, \tilde{U})$. Thus

$$y(t) = h(\phi(R(\tilde{Y}(t), \tilde{U}(t))))$$
$$u(t) = \psi(R(\tilde{Y}(t), \tilde{U}(t)), \tilde{U}(t))$$

define the transformations ϕ^e and ψ^e. Similarly we get the transformations $\tilde{\phi}^e$ and $\tilde{\psi}^e$, which gives external dynamic equivalence of Σ and $\tilde{\Sigma}$.

Now assume that Σ and $\tilde{\Sigma}$ are externally dynamically equivalent. Thus external trajectories of both systems are related by equations

$$y(t) = \phi^e(\tilde{Y}(t), \tilde{U}(t))$$
$$u(t) = \psi^e(\tilde{Y}(t), \tilde{U}(t)).$$

They can be extended to jets

$$Y(t) = \Phi^e(\tilde{Y}(t), \tilde{U}(t))$$
$$U(t) = \Psi^e(\tilde{Y}(t), \tilde{U}(t)).$$

Again from uniform observability, $x = \varrho(Y, U)$, so

$$x(t) = \varrho(\Phi^e(\tilde{Y}(t), \tilde{U}(t)), \Psi^e(\tilde{Y}(t), \tilde{U}(t))).$$

However $Y = \Gamma(X, U)$ and $U = S(X)$, so we finally obtain

$$x(t) = \phi(\tilde{X}(t))$$
$$u(t) = \psi(\tilde{X}(t), \tilde{U}(t)).$$

for some finitely presented maps ϕ and ψ. Similarly for $\tilde{x}$ and $\tilde{u}$. ∎

We can state now the main result of this paper.

Theorem 5.7. *Two uniformly observable systems Σ and $\tilde{\Sigma}$ are externally dynamically equivalent if and only if their delta universes $\mathcal{U}_\Sigma$ and $\mathcal{U}_{\tilde{\Sigma}}$ are isomorphic.*

Proof. The theorem follows from Corollary 5.5. and Proposition 5.6.. ∎

Example 5.8. *The concept of external dynamic equivalence of systems on time scales may be used to give a unified setting for external dynamic linearization. In [21] and [19] it was shown that a continuous-time (discrete-time) system is externally dynamically linearizable if and only if its differential output universe (difference output universe, respectively) is free. To have this result on an arbitrary homogeneous time scale, it is enough to replace differential and difference output universes by the delta universe of the system on the time scale.*

6. CONCLUSIONS AND FUTURE WORKS.

6.1. Conclusions. The theory of control systems on time scales allows to use a common language for continuous-time and discrete-time systems. The concept of external dynamic equivalence of control systems with output unifies the earlier notions introduced separately for continuous-time and discrete-time systems. It was shown that two uniformly observable systems on a time scale are externally dynamically equivalent if and only if their delta universes are isomorphic. This extends corresponding results for continuous-time and discrete-time systems, where differential and difference output universes were used instead of the delta universe. The language introduced here allows to study external dynamic linearizations for systems on time scales.

6.2. Future Works. The main assumption on the systems in this paper is uniform observability. It was shown in [21] and [3] that for continuous-time and discrete-time systems the criterion of external dynamic equivalence holds without this assumption. Thus one can hope that also for systems on a time scale observability can be dropped. This, however, requires a thorough study of observability of systems on time scales. A more serious challenge is to extend the result to nonhomogeneous time scales. In this case the delta operator may depend on time and this requires a new language. Another interesting extension should lead to modification of the concept of dynamic equivalence by allowing a change of time or, more generally, time scale.

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