INPUT-OUTPUT EQUIVALENCE TRANSFORMATIONS FOR DISCRETE-TIME NONLINEAR SYSTEMS

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Abstract: The paper characterizes the input-output (i/o) equivalence transformations for discrete-time nonlinear system described by the set of nonlinear higher order difference equations in the inputs and the outputs. It has been demonstrated that applying the i/o equivalence transformation to the set of equations $\varphi = 0$ amounts to multiplying φ from left by the unimodular matrix $U(\delta)$, whose entries are non-commutative polynomials in the forward-shift operator δ. Then it has been proved that using the i/o equivalence transformations, the set of equations can be transformed into the row-reduced form, whenever the set of equations is well-defined. Finally, the constructive algorithm is provided for calculation of the transformation, which extends the corresponding transformation for linear systems and is directly implementable via the computer algebra systems like *Mathematica* or *Maple*. *Copyright© 2004 IFAC.*

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1. INTRODUCTION

One of the central themes in system theory is the problem of representing a system in a form which is convenient for the particular purpose one has in mind, and of transforming one representation into another. Particularly, for linear systems, it is well known that an arbitrary set of higher order input-output (i/o) difference equations can be always transformed into an *equivalent* set of equations having a row-reduced (row-proper) form, whenever the system is nonsingular.

The main purpose of this paper is to characterize the i/o equivalence transformations for discrete-time nonlinear system described by the set of analytic higher order difference equations in the inputs and the outputs, and to transform the system equations via i/o equivalence transformations into an equivalent, but row-reduced form. Our interest in this form originates from the fact that this is a necessary step for realization of the i/o difference equations in the classical state space form. Recall that the realization procedure in (Kotta *et al.*, 2001) requires the possibility to associate with the i/o equations an extended state-space system. Once the set of nonlinear higher order

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difference equations is in the row- and column-reduced form, it is extremely easy to write down the associated extended system. So, the row- and column-reduced forms of the set of high order i/o difference equations will be instrumental to all the further developments of multi-input multi-output realization problem.

Note that the preliminary results on transformation of the i/o difference equations into a row-reduced form have been reported in (Kotta, 1999). This paper differs from (Kotta, 1999) in many aspects. First, we *characterize* the class of i/o equivalence transformations. Second, the paper (Kotta, 1999) only proves the existence of transformations, whereas we give the constructive algorithm that transforms the system into the row-reduced form. Two distinctive features of our polynomial matrix approach are the following.

- The algorithm that relies on application of unimodular matrices remarkably coincides with the corresponding result for the linear systems.
- The algorithm can be directly implemented via computer algebra systems like *Mathematica* or *Maple*.

The basic difference is that the polynomial entries of the unimodular matrix belong to non-commutative polynomial ring.

2. PROBLEM STATEMENT AND PRELIMINARIES

Consider a discrete-time multi-input multi-output nonlinear system Σ described by the set of higher order implicit input-output difference equations

$$\varphi_i(y(t), \ldots, y(t+n), u(t), \ldots, u(t+n)) = 0, \quad i = 1, \ldots, p \tag{1}$$

where $u \in I\!R^m$ is the input variable, $y \in I\!R^p$ is the output variable and φ_i is a real analytic function defined on $I\!R^{(n+1)(p+m)}$. The sequence $\{u(t), y(t); t \geq 0\}$ is called a solution of (1), if for any $t \geq 0$, $u(t), y(t)$ satisfy equation (1). We assume that the system of equations (1) is well-defined, i. e. that the output sequence $y(\cdot)$ of the solution is uniquely defined by the input sequence under the fixed initial conditions.

A system may be described by different sets of equations. This leads to the problem of equivalence.

Definition 1. Two discrete-time control systems of the form (1) are called input-output equivalent [3] if their solutions are the same.

[3] The i/o equivalence notion is different from transfer equivalence, see (Kotta *et al.*, 2001)

An equivalence transformation for system (1) is a transformation of the set of equations $\varphi_i = 0$, $i = 1, \ldots, p$ into another set of equations $\tilde{\varphi}_I = 0$, $I = 1, \ldots, P$ of the form (1) that is input-output equivalent to the original system.

2.1 The row-reduced form of the set of i/o difference equations

Define for any $i \in \{1, \ldots, p\}$ the row degree of $\varphi_i(\cdot)$ in the output y, σ_i, as the largest integer ($\leq n$) such that

$$\frac{\partial \varphi_i}{\partial y_j(t+\sigma_i)} \neq 0 \tag{2}$$

for some $j \in \{1, \ldots, p\}$, i. e. σ_i is the highest time-shift of the output component appearing non-trivially in φ_i (the highest power of the shift operator δ applied to y, appearing in φ_i). If σ_i is not defined, i. e. if φ_i does not depend on y, then we set $\sigma_i = -1$. Now define the $p \times p$ matrix $A_{hr}(y(t), \ldots, y(t+n), u(t), \ldots, u(t+n))$ as the matrix with (i,j)-th element given by (2).

Definition 2. The pair of constant values (y_e, u_e) is called the equilibrium point of the system (1) if (y_e, u_e) satisfies the equalities $\varphi_i(y_e, \ldots, y_e, u_e, \ldots, u_e) = 0$, $i = 1, \ldots, p$.

Definition 3. (Kotta, 1999) The set of the higher order i/o difference equations (1) is said to be locally row-reduced if for all $(y(t), \ldots, y(t+n), u(t), \ldots, u(t+n))$ around the equilibrium point $(0,0)$, rank $A_{hr}(\cdot) = p$.

In the linear case being in a row-reduced form means that the polynomial matrix, defining the set of equations (1), is row-proper. The aim of this paper is to characterize a class of i/o equivalence transformations, acting on (1) and to give an algorithm that transforms the system into the equivalent, but the row-reduced form whenever the system of equations is well-defined.

2.2 Introduction of the forward-shift operator δ for the system of implicit input-output equations

Since the equations of the form (1) are implicit, the standard way (Aranda-Bricaire *et al.*, 1996) to define the forward-shift operator δ, associated to the system, cannot be used.

Let $y(t) = (y_1(t), \ldots, y_p(t))$ and $u(t) = (u_1(t), \ldots, u_p(t))$, $t \geq 0$. Denote $y = y(0)$, $u = u(0)$. Let $Y = (y, y(1), \ldots,)$ and $U = (u, u(1), u(2), \ldots)$. We think of components of Y and U as independent variables. Let Mer_0 be the set of all meromorphic functions with real values depending on finitely many elements of Y and U and well defined at $Y = 0$, $U = 0$. Mer_0 is a ring with addition and multiplication. Let $\delta : \mathrm{Mer}_0 \to \mathrm{Mer}_0$ be defined as follows: $\delta y_i(t) = y_i(t+1)$, $\delta u_i(t) = u_i(t+1)$ and for $\varphi \in \mathrm{Mer}_0$, $(\delta\varphi)(Y,U) := \varphi(\delta Y, \delta U)$, where $\delta Y = (y(1), y(2), \ldots)$ and $\delta U = (u(1), u(2), \ldots)$. Then Mer_0 is a difference ring with a difference operator δ.

Let $\Phi = \{\varphi_1, \ldots, \varphi_p\}$ be a finite subset of Mer_0. Φ may be interpreted as a system of implicit input-output equations. Let $< \Phi >$ denote the difference ideal of Mer_0, generated by Φ. Then $\delta < \Phi > \subset < \Phi >$ but , in general, $< \Phi > \not\subset \delta < \Phi >$.

Let $R = \mathrm{Mer}_0 / < \Phi >$ be the quotient ring. It consists of cosets $\bar{\varphi} = \varphi + < \Phi >$ for $\varphi \in \mathrm{Mer}_0$. We define "+" and "·" on R by $\overline{\varphi} + \overline{\psi} := \overline{\varphi + \psi}$ and $\overline{\varphi} \cdot \overline{\psi} := \overline{\varphi \cdot \psi}$. These definitions do not depend on the choice of a representative in a coset. The set R with "+" and "·" is a commutative ring whose "0" element is just $< \Phi >$. In particular $\overline{\varphi}_i = 0$, for $i = 1, \ldots, p$.

Now we can define δ on R as follows: $\delta\overline{\varphi} = \overline{\delta\varphi}$. This again is well defined, for if $\overline{\varphi} = \overline{\psi}$, then $\varphi + < \Phi > = \psi + < \Phi >$. Since $\delta < \Phi > \subset < \Phi >$ and $\delta < \Phi > + < \Phi > = < \Phi >$, then $\delta\varphi + \delta < \Phi > = \delta\psi + \delta < \Phi >$. We often write $\delta = \delta_\Phi$ to indicate that the difference operator R is related to the system Φ. The operator δ is one-to-one iff $\delta\varphi \in < \Phi >$ implies $\varphi \in < \Phi >$. This is true if the functions in Φ cannot be shifted back, i. e. if in each equation some $u_i(t)$ or $y_j(t)$ appears. We assume that this holds for system (1).

2.3 Non-commutative ring of polynomials and polynomial matrix description of nonlinear system

Let $\mathcal{K}$ denote the field of meromorphic functions in a finite number of the variables of Y and U. If δ is one-to-one, the pair $(\mathcal{K}, \delta)$ is a difference field. Over the field $\mathcal{K}$ one can define a difference vector space, $\mathcal{E} := \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\varphi \mid \varphi \in \mathcal{K}\}$. The operator $\delta : \mathcal{K} \to \mathcal{K}$ induces a forward-shift operator $\delta : \mathcal{E} \to \mathcal{E}$ by $\sum_i a_i \mathrm{d}\varphi_i \to \sum_i (\delta a_i)\mathrm{d}(\delta\varphi_i)$, $\quad a_i, \varphi_i \in \mathcal{K}$.

The field $\mathcal{K}$ and the shift operator δ induce a ring of polynomials in the shift operator δ, denoted by $\mathcal{K}[\delta]$. A polynomial $p(\delta) \in \mathcal{K}[\delta]$ is written as

$$p(\delta) = a_m\delta^m + a_{m-1}\delta^{m-1} + \ldots + a_1\delta + a_0, \quad (3)$$

where $a_i \in \mathcal{K}$ for $0 \le i \le m$. Moreover, $\deg[p_1(\delta) \cdot p_2(\delta)] = \deg p_1(\delta) + \deg p_2(\delta)$. Each polynomial $p(\delta) \in \mathcal{K}[\delta]$ is a mapping of $\mathcal{E}$ into itself. To evaluate $p(\delta)$ at any $\omega \in \mathcal{E}$, note that for nonlinear systems, the coefficients a_i in (3) are no more constants as in the linear case but meromorphic functions in a finite number of variables $y_s(t), \ldots, y_s(t+N)$, $s = 1, \ldots, p$ and $u_k(t)$, $k = 1, \ldots, m$, $t \ge 0$ with N large enough. The latter implies that an element $a \in \mathcal{K}$ does not commute with the shift operator δ, i. e. $a \cdot \delta \ne \delta \cdot a$ as $a \cdot \delta y_s(t) = a y_s(t+1)$ is not equal to $\delta \cdot a y_s(t) = \delta a \cdot y_s(t+1)$. Since the multiplication between the shift operator δ and an element $a \in \mathcal{K}$ is not commutative and can be defined by the following rule

$$\delta \cdot a = \delta a \cdot \delta, \quad (4)$$

(so for example $(p\delta^n)(q\delta^n) = p\delta^n(q)\delta^{n+m}$), the ring $\mathcal{K}[\delta]$ thus defined is a non-commutative ring. If the multiplication is defined by (4), the non-commutative ring $\mathcal{K}[\delta]$ is called the *twisted polynomial ring* twisted by δ and it is proved to satisfy both the left and right Øre conditions, i. e. to be Øre ring (Farb and Dennis, 1993). If the non-commutative ring satisfies the Øre condition, one can construct the division ring of fractions, a process exactly like that of constructing the field of rational numbers from the ring of integers.

Note that all the other algebraic operations in the ring satisfy the operations in the field (of meromorphic functions) $\mathcal{K}$. The ring of polynomicals $K[\delta]$ has all the properties of field except for the inverse, i. e. the inverse of a polynomial of degree one or more is not a polynomial.

We now represent the nonlinear system (1) in terms of two polynomial matrices, with the polynomials in the Øre ring (twisted polynomial ring). For that we apply the differential operation to (1) to obtain

$$\sum_{s=1}^{p}\sum_{j=0}^{n} \frac{\partial\varphi_i}{\partial y_s(t+j)} \mathrm{d}y_s(t+j) = \sum_{k=1}^{m}\sum_{r=0}^{n} \frac{\partial\varphi_i}{\partial u_k(t+r)} \mathrm{d}u_k(t+r), \quad i = 1, \ldots, p. \quad (5)$$

Since $\mathrm{d}y_s(t+j) = \delta^j \mathrm{d}y_s(t)$, $\mathrm{d}u_k(t+r) = \delta^r \mathrm{d}u_k(t)$, we can rewrite (5) as

$$P(\delta)\mathrm{d}y(t) = Q(\delta)\mathrm{d}u(t) \quad (6)$$

where $P(\delta)$ and $Q(\delta)$ are $p \times p$ and $p \times m$-dimensional matrices respectively, whose elements $p_{ij}, q_{ij} \in \mathcal{K}[\delta]$:

$$p_{is}(\delta) = \sum_{j=0}^{n} \frac{\partial\varphi_i}{\partial y_s(t+j)} \delta^j$$

$$q_{ik}(\delta) = -\sum_{r=0}^{s} \frac{\partial\varphi_i}{\partial u_k(t+r)} \delta^r$$

and $\mathrm{d}y(t) = \mathrm{d}[y_1(t), \ldots, \mathrm{d}y_p(t)]^T$, $\mathrm{d}u(t) = [\mathrm{d}u_1(t), \ldots, \mathrm{d}u_m(t)]^T$.

3. POLYNOMIAL MATRICES WITH ELEMENTS IN $\mathcal{K}[\delta]$

We now consider a class of matrices $P(\delta)$ whose elements are polynomials $p(\delta) \in \mathcal{K}[\delta]$ of finite, but unbounded degree. We write $\mathcal{K}^{p\times q}[\delta]$ for the set of $p \times q$ matrices with entries in $\mathcal{K}[\delta]$. These polynomial matrices differ from polynomial matrices $\mathbb{R}^{p\times q}[\delta]$, known from the linear system theory in a fundamental way – the polynomial entries of the matrices are non-commutative. The purpose of this section is to show that like in the linear case where the polynomials have real coefficients, the polynomial matrix in $\mathcal{K}^{p\times q}[\delta]$ can be transformed by a sequence of elementary row operations into the row proper form. This result allows us to transform the set of i/o difference equations into an equivalent system in the row proper form.

Definition 4. The following three elementary *row operations* $E(\delta)$ on the polynomial matrix $P(\delta)$ are defined

(1) Interchange of rows i and j.
(2) Multiplication of row i by nonzero scalar in $\mathcal{K}$.
(3) Replacement of row i by itself plus any polynomial multiplied by any other row j.

Definition 5. A **unimodular matrix** $U(\delta)$ is defined as any square matrix which can be obtained from the identity matrix $\mathcal{I}$ by a finite number of elementary row operations on $\mathcal{I}$: $U(\delta) = E_N(\delta)E_{N-1}(\delta)\dots E_1(\delta)\mathcal{I}$.

Any sequence of elementary row operations on $P(\delta)$ is equivalent to premultiplication (left multiplication) of $P(\delta)$ by an appropriate unimodular matrix $U(\delta)$. The inverse matrix of the unimodular matrix can be defined as $[U(\delta)]^{-1} = [E_1(\delta)]^{-1}[E_2(\delta)]^{-1}\dots[E_N(\delta)]^{-1}$ because for each elementary operation the inverse operation can be easily defined.

Definition 6. Two polynomial matrices $P(\delta)$ and $\hat{P}(\delta)$ will be called row equivalent iff one of them can be obtained from the other by a sequence of elementary row operations.

$P(\delta)$ is thus row equivalent to $\hat{P}(\delta)$ if and only if $P(\delta) = U(\delta)\hat{P}(\delta)$ where $U(\delta)$ is unimodular matrix.

For a nonzero polynomial row $p(\delta) \in \mathcal{K}^{1\times p}[\delta]$, we define its degree as the exponent of the highest power in δ present in $p(\delta)$. We denote this degree by $\deg p(\delta)$ and it follows that $\deg p(\delta) \geq 0$. If $p(\delta) \equiv 0$ we define $\deg p(\delta) = -\infty$. Let $P(\delta)$ be a polynomial matrix in $\mathcal{K}^{p\times q}[\delta]$ with rows $P_1(\delta),\dots,P_p(\delta) \in \mathcal{K}^{1\times q}[\delta]$. We denote $\deg P_i(\delta) = \sigma_i$ for $i = 1,\dots,p$, and write $P_i(\delta) = P_{io}\delta^{\sigma_i} + P_{i1}\delta^{\sigma_i-1} + \dots + P_{i\sigma_i}$ with P_{ij} being a $1\times q-$ dimensional row vector of functions in $\mathcal{K}$ for $j = 0,\dots,\sigma_i$. The scalar matrix with elements in $\mathcal{K}$ consisting of the coefficients of the highest degree δ terms in each row of $P(\delta)$, $P_{hr} = [P_{10}^T,\dots,P_{po}^T]^T$ is called the leading row coefficient matrix of $P(\delta)$.

The **degree** of a nonzero polynomial matrix $P(\delta)$ is equal to the degree of the polynomial element of highest degree in $P(\delta)$.

Definition 7. A $p\times m$ polynomial matrix, $P(\delta)$, will be called **row proper** if and only if its its leading row coefficient matrix P_{hr} has full row rank over $\mathcal{K}$.

Theorem 8. Any polynomial matrix $P(\delta)$, corresponding to the well-defined system of equations (1) is row equivalent to a row proper matrix, i. e. one can always transform $P(\delta)$ via elementary row operations to row proper form.

Proof. If $P(\delta)$ is not row proper, then we can bring it into a row proper form. To see how this can be done, assume that the row degrees σ_i of $P(\delta)$ are ordered in a nondecreasing way, i. e. $\sigma_i \leq \sigma_{i+1}, 1 \leq i < p$. Since $P(\delta)$ is not row proper, the rows in P_{hr} are linearly dependent. Therefore, there exists an index $k < p$ such that the $(k+1)$th row $P_{hr,k+1}$ in P_{hr} can be written as a linear combination of the rows $P_{hr,1},\dots,P_{hr,k}$ in P_{hr}. Consequently, there are coefficients $\alpha_i \in \mathcal{K}$, $i = 1,\dots,k$ such that $P_{hr,k+1} + \sum_{i=1}^{k}\alpha_i P_{hr,i} = 0$. Given these coefficients, we can replace row $P_{k+1}(\delta)$ of $P(\delta)$ by a a linear combination of the $(k+1)$th row and the forward time-shifts (up to certain order) of the 1st up to kth rows,

$$P_{k+1}(\delta) := P_{k+1}(\delta) + \sum_{i=1}^{k}\alpha_i\delta^{\mathrm{d}_i}P_i(\delta), \qquad (7)$$

where $\mathrm{d}_i = \sigma_{i+1} - \sigma_i$ for $i = 1,2,\dots,k$. The transformation (7) corresponds to the following multiplication $E_1(\delta)P(\delta)$ where

$$E_1(\delta) = \begin{bmatrix} 1 & 0 & \dots & 0 & 0\;0\dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots\;\vdots\;\ddots\;\vdots \\ 0 & 0 & \dots & 1 & 0\;0\dots 0 \\ \alpha_1\delta^{\mathrm{d}_1} & \alpha_2\delta^{\mathrm{d}_2} & \dots & \alpha_k\delta^{\mathrm{d}_k} & 1\;0\dots 0 \\ 0 & 0 & \dots & 0 & 0\;1\dots 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots\;\vdots\;\ddots\;\vdots \\ 0 & 0 & \dots & 0 & 0\;0\dots 1 \end{bmatrix}$$

is the identity matrix where the $(k+1)$th row is replaced by $(\alpha_1\delta^{\mathrm{d}_1},\alpha_2\delta^{\mathrm{d}_2},\dots,\alpha_k\delta^{\mathrm{d}_k},1,\dots,0,\dots,0)$,

i. e. $E_1(\delta)$ defines the elementary equivalence transformation.

It easily follows that the row degree of row $P_{k+1}(\delta)$ now has become less than σ_{k+1}. The other rows of $P(\delta)$ and their degrees remain unchanged. Hence, the *sum* of the row degrees of $P(\delta)$ has been decreased by at least one.

We can now check whether or not $P(\delta)$ is row proper. If this is the case we can stop, and we can immediately obtain its row degrees. If $P(\delta)$ is not row proper, we must repeat the above procedure. Since in each step the sum of the row degrees decreases, but never can become less than zero, this process is guaranteed to end. It either ends with a row proper matrix, or the system of equations is inconsistent or trivially satisfied.

4. I/O EQUIVALENCE TRANSFORMATIONS

The purpose of this section is to demonstrate that i/o equivalence transformations are linear transformations with coefficients in Mer_0.

First observe that the set of solutions (zeros) of a difference system of the form (1), $\varphi_i = 0$, $i = 1, \dots, p$, and the set of zeros of the difference ideal generated by φ_i, $i = 1, \dots, p$, (in the ring Mer_0) are the same. Thus two input-output difference systems of the form (1), $\varphi_i = 0$, $i = 1, \dots, p$, and $\tilde{\varphi}_I = 0$, $I = 1, \dots, P$, are input-output equivalent, if their difference ideals, $<\varphi_1, \dots, \varphi_p>$ and $<\tilde{\varphi}_1, \dots, \tilde{\varphi}_P>$ respectively, are equal. Unfortunately, in general, this implication cannot be reversed.

Let $Z(<\varphi_1, \dots, \varphi_p>)$ denote the set of zeros of the difference ideal $<\varphi_1, \dots, \varphi_p>$ and let $I(Z(<\varphi_1, \dots, \varphi_p>)) := \{f \in \mathrm{Mer}_0 : f(Z((\varphi_1, \dots, \varphi_p))) = 0\}$ denote the *zero-ideal* of this set.

Thus we have: two input-output difference systems of the form (1), $\varphi_i = 0$, $i = 1, \dots, p$, and $\tilde{\varphi}_I = 0$, $I = 1, \dots, P$, are input-output equivalent, if and only if $Z <\varphi_1, \dots, \varphi_p> = Z <\tilde{\varphi}_1, \dots, \tilde{\varphi}_P>$.

In general $I(Z(<\varphi_1, \dots, \varphi_p>)) \supseteq <\varphi_1, \dots, \varphi_p>$. We make an **Assumption 2:**

$$<\varphi_1, \dots, \varphi_p> = I(Z(<\varphi_1, \dots, \varphi_p>))$$

for all difference systems we consider here. The Assumption 2 roughly means that system $\varphi_i = 0$, $i = 0, \dots, p$ cannot be simplified. Under Assumption 2, two input-output difference systems of the form (1), $\varphi_i = 0$, $i = 1, \dots, p$, and $\tilde{\varphi}_I = 0$, $I = 1, \dots, P$, are input-output equivalent, if and only if

$$<\varphi_1, \dots, \varphi_p> = <\tilde{\varphi}_1, \dots, \tilde{\varphi}_P>. \tag{8}$$

The relationship (8) again means that since $\tilde{\varphi}_i \in <\varphi_1, \dots, \varphi_p>$, we can write, because of the definition of difference ideal, that

$$\tilde{\varphi}_i = \sum_{k=0}^{N} \sum_{j=1}^{p} \alpha_{ijk} \delta^k \varphi_j, \tag{9}$$

or shortly $\tilde{\varphi} = A(\delta)\varphi$, where $\alpha_{ijk} \subset Mer_0$, i. e. α_{ijk} are meromorphic functions such that their denominators are not zero at $Y = 0$, $U = 0$.

In the similar manner, because of (8), we can express φ_i's via $\tilde{\varphi}_j$'s:

$$\varphi_i = \sum_{k=0}^{n} \sum_{j=1}^{P} \beta_{ijk} \delta^k \tilde{\varphi}_j. \tag{10}$$

or $\varphi = B(\delta)\tilde{\varphi}$.

Thus the polynomial matrix $A(\delta)$ must have a polynomial inverse $A^{-1}(\delta) = B(\delta)$. This is possible only if $A(\delta)$ is a unimodular matrix, or equivalently, it is a product of elementary matrices $E(\delta)$.

Note that the i/o equivalence transformations are specified by the property that $A(\delta)$ is a unimodular matrix.

5. TRANSFORMATION OF THE SET OF DIFFERENTIALS OF INPUT-OUTPUT EQUATIONS INTO THE ROW-REDUCED FORM

In this section we develop a method for transforming the set of nonlinear input-output difference equations into a row proper form.

Consider the set of i/o difference equations (1), i. e.

$$\varphi_i(\cdot) = 0, \quad i = 1, \dots, p \tag{11}$$

If we apply the differential operation d to (11), we obtain, as in (6):

$$P(\delta)\mathrm{d}y(t) = Q(\delta)\mathrm{d}u(t). \tag{12}$$

In Section 3 it has been demonstrated that there exists a unimodular matrix $U(\delta)$ which transforms the matrix $P(\delta)$ into a row-proper form $\tilde{P}(\delta)$: $\tilde{P}(\delta) = U(\delta)P(\delta)$. Multiplying the matrix $Q(\delta)$ in (12) by the same unimodular matrix $U(\delta)$, we reach the equivalent, but row-proper form of (12), i. e.

$$U(\delta)P(\delta)\mathrm{d}y(t) - U(\delta)Q(\delta)\mathrm{d}u(t) = 0. \tag{13}$$

The one-forms defined by (13) are, in general, not integrable; in particular this is the case if the coefficients α_k of polynomial entries in $U(\delta)$ are not real, but belong to the field $\mathcal{K}$. In that case, to make (13) integrable, we have to add to it $\mathrm{d}U(\delta)\cdot\varphi$ to get

$$\begin{aligned}\mathrm{d}\tilde{\varphi} = U(\delta)[P(\delta)\vdots - Q(\delta)]\begin{bmatrix}\mathrm{d}y(t)\\ \mathrm{d}u(t)\end{bmatrix} + \\ +\mathrm{d}U(\delta)\cdot\varphi = \mathrm{d}[U(\delta)\varphi] = 0\end{aligned} \tag{14}$$

where in $\mathrm{d}[U(\delta)]$ the differentiation is done elementwise.

Integration of (14) gives us the equivalent, but row-reduced set of i/o difference equations $\tilde{\varphi} = U(\delta)\varphi$. Note that

$$\mathrm{d}E_1(\delta) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \mathrm{d}a_1\delta^{\mathrm{d}_1} & \mathrm{d}a_2\delta^{\mathrm{d}_2} & \dots & \mathrm{d}a_k\delta^{\mathrm{d}_k} & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and $\mathrm{d}U(\delta) = \mathrm{d}[E_N(\delta)\dots E_1(\delta)] = \mathrm{d}E_N(\delta) + \dots + \mathrm{d}E_1(\delta)$.

Example. Consider the system

$$\begin{aligned}\varphi_1 &= y_1(t+2) - u_1(t) = 0\\ \varphi_2 &= -y_1^2(t+3)y_1(t)\\ &+ 2y_1(t+3)y_1(t)u_1(t+1)\\ &- u_1^2(t+1)y_1(t) + y_2(t+1) - u_2(t) = 0\end{aligned} \tag{15}$$

The matrix $P(\delta)$ for system (15) is

$$P(\delta) = \begin{bmatrix} \delta^2 & 0 \\ -2y_1(t)[y_1(t+3) - u_1(t+1)]\delta^3 - \\ -[y_1(t+3) - u_1(t+1)]^2 & \delta \end{bmatrix}$$

which is not in the row-reduced form. To transform $P(\delta)$ into the row reduced form we will multiply it by the unimodular matrix

$$U(\delta) = \begin{bmatrix} 1 & 0 \\ 2y_1(t)[y_1(t+3) - u_1(t+1)]\delta & 1 \end{bmatrix}$$

which yields the system $\tilde{P}(\delta)\mathrm{d}y(t) = \tilde{Q}(\delta)\mathrm{d}u(t)$ where

$$\tilde{P}(\delta) = \begin{bmatrix} \delta^2 & 0 \\ -[y_1(t+3) - u_1(t+1)]^2 & \delta \end{bmatrix},$$

$$\tilde{Q}(\delta) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The first one-form $\mathrm{d}y(t+2) - \mathrm{d}u_1(t)$ is integrable but the second

$$-[y_1(t+3) - u_1(t+1)]^2\mathrm{d}y_1(t) + \mathrm{d}y_2(t) - \mathrm{d}u_2(t)$$

is not. If we add to $\tilde{P}(\delta)\mathrm{d}y(t) - \tilde{Q}(\delta)\mathrm{d}u(t)$

$$\mathrm{d}U(\delta)\cdot\varphi =$$

$$= \begin{bmatrix} 0 & 0 \\ -\mathrm{d}[2y_1(t)[y_1(t+3) - u_1(t+1)]]\delta & 0 \end{bmatrix}\begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix},$$

the second one-form reads as $\mathrm{d}\tilde{\varphi} = \mathrm{d}y_2(t+1) - \mathrm{d}u_2(t) + \mathrm{d}[y_1(t)(y_1(t+3) - u_1(t+1))^2]$. From the above we get, since the second term is equal to zero, $\tilde{\varphi}_2 = y_2(t+1) - u_2(t) = 0$.

6. CONCLUDING REMARKS

The paper provides the characterization of the i/o equivalence transformations for discrete-time nonlinear higher order difference equations in the inputs and outputs. It has been demonstrated that applying the i/o equivalence transformation to the set of equations $\varphi = 0$ amounts to multiplying φ form left by the unimodular matrix $U(\delta)$, whose entries are non-commutative polynomials in the forward-shift operator δ. Then, it has been proved that using the i/o equivalence transformations, the set of equations can be transformed into the row-reduced form, whenever the set of equations is well-defined. Finally, the constructive algorithm is provided for calculation of the equivalence transformation which extends the corresponding transformation for linear systems and is directly implementable via the computer algebra systems like *Mathematica* or *Maple*.

The problem of transforming the set of i/o difference equations in a doubly-reduced (i. e. both row- and column-reduced) form is the topic of the future paper. Note that the doubly-reduced form is instrumental in the solution of the realization problem of the i/o difference equations into the classical state space form.

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