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ON FAMILIES OF MULTIOBSERVERS FOR NONLINEAR SYSTEMS

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Abstract. A construction of a family of multiobservers for locally observable systems is introduced. Existence of continuous multiobserver can only be claimed. The dynamics of multiobserver system gives an estimation of the successive derivatives of the output based upon the knowledge of the original system's output and its input. The multiobserver's output is a multivalued mapping which estimates the whole class of states that are indistinguishable from the current state of the system.

Key Words: observers, observability, multivalued mapping, nonlinear systems, continuous systems.

1 INTRODUCTION

Observing states of nonlinear systems is not as easy as in the linear case, where as soon as the system is observable, an observer can be designed. Because local observability at every point for nonlinear systems is much easer to check than (global) observability so a kind of observers, which are called multiobservers, will be constructed for locally observable systems. A construction of a multiobserver for a nonlinear analytic system have been proposed in [3]. The main ideas of Jouan and Gauthier [5] are followed here. If local observability of the system on a compact set is assumed, higher derivatives of the output can be computed as continuous functions of lower derivatives. This function depends on the output, the input and their time derivatives. As one of such functions appears in the dynamics of the (multi)observer, the dynamics is only continuous. The multiobserver's output is multivalued mapping which gives simultaneously finitely many estimations of the state of the original system. They estimate the state of the system modulo the indistinguishability relation. The function, which describes a relation between the derivatives of the output, is not always locally Lipschitz. Lipschitz property would allow to obtain exponential and unique estimate of the time derivatives of the output y, so our goal is to improve the (multi)observer to receive a locally Lipschitz relation. It is assumed that there is a family of continuous and Lipschitz functions with some Lipschitz constant. Then one function from the family is taken to the dynamics of the (multi)observer. In this way the solution of the (multi)observer dynamics approximates the successive derivatives of the original system's output with an error which depends on the function that will be chosen. In [6] such a family of multiobservers was constructed for a system without control. In this paper we extend the ideas and results of [6] to the controlled case.

2 BASIC DEFINITIONS AND FACTS

Let us consider a nonlinear analytic system Σ on $\Omega \subset \mathbb{R}^n$:

$$\dot{x} = f(x,u) \tag{1}$$

$$y = h(x) \tag{2}$$

The output $y(t)$ and the input $u(t)$ are assumed to be scalar for simplicity. The vector field f and the function h are assumed to be real analytic.

Two points x_1 and x_2 in $\mathbb{R}^n$ are *indistinguishable* if

$$h(x(t,x_1,u)) = h(x(t,x_2,u)) \tag{3}$$

for every control u and every $t \geq 0$ for which both sides of the equation 3 are defined. Here $x(t, x_0, u)$ denotes the trajectory of f starting at $x(0) = x_0$, corresponding to control u and evaluated at time t. Otherwise x_1 and x_2 are called *distinguishable*. Indistinguishability is an equivalence relation so the state space can be decomposed into disjoint indistinguishability classes.

The system Σ is *observable* if any two distinct points are distinguishable.

The system Σ is *locally observable* at x_0 if there is a neighborhood U of x_0 such that for every $x \in U$, x and x_0 are distinguishable. Σ is *locally observable* if it is locally observable at every point. Necessary and sufficient conditions for local observability were developed in [1, 2].

Assume that Σ is locally observable on a compact set $\Omega \subset \mathbb{R}^n$. Then any indistinguishability class in Ω is finite and there is a common bound on the number of elements in a class.

Let φ depend on $x \in \mathbb{R}^n, u \in \mathbb{R}$ and a finite jet of u: $u', u'', \dots, u^{(k)}$. Then we define

$L_f\varphi(x, u, \dots, u^{(k+1)}) = \frac{\partial\varphi(x,\dots,u^{(k)})}{\partial x} f(x, u) + \sum_{i=0}^{k} \frac{\partial\varphi(x,\dots,u^{(k)})}{\partial u^{(i)}} u^{(i+1)}$.

An important fact in construction of observers is the following proposition whose proof is an extension of the proof of a result in [6] to the controlled case.

Proposition 2.1. *If the system Σ is locally observable on compact set Ω then there is $N \in \mathbb{N}$ and there is a continuous function $\varphi_N : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ such that*

$$L_f^N h = \varphi_N(h, \dots, L_f^{N-1}h, u, \dot{u}, \dots, u^{(N-1)}).$$

When the continuous function φ_N which is got from Proposition 2.1 is Lipschitz on compact subsets then a multiobserver for locally observable systems can be constructed [3]. This construction is based on the idea of Jouan and Gauthier [5].

3 MULTIOBSERVERS AND THEIR CONSTRUCTION

The system $\mathcal{S}$

$$\dot{z} = F(z, y, u, \dots u^{(N-1)}) \tag{4}$$

$$\hat{x} = g(z, u, \dots u^{(N-2)}) \tag{5}$$

is called *a multiobserver of the system Σ on the compact subset* $\Omega \subset \mathbb{R}^n$ if:

(1) the inputs y and u are respectively the output and the input of the system Σ,
(2) g is a multivalued map whose values are finite subsets of $\mathbb{R}^n$,
(3) $\lim_{t\to+\infty} d(x(t); \hat{x}(t)) = 0$,
where $d(x_0; A) := \min_{x\in A} |x_0 - x|$ is a distance from x_0 to A, $|\cdot|$ is the Euclidean norm, F and g are assumed to be continuous in appropriate topologies.

Equation (4) is called the *dynamical* part of the multiobserver $\mathcal{S}$, while (5) is the *static* part. The multiobserver $\mathcal{S}$ will be *continuous* (*smooth*) if F and g are continuous (smooth) in appropriate topologies.

The multiobserver $\mathcal{S}$ of the system Σ (on the compact subset $\Omega \subset \mathbb{R}^n$) is a system whose input is the output of the system Σ and whose output is a multivalued map which estimates the whole class of states that are indistinguishable from the current state of the system Σ. The state of the multiobserver $\mathcal{S}$ approximates the time derivatives of the output y of Σ based upon the knowledge of the output $y = h(x)$ and the input u. To achieve this let's assume that higher order derivatives of the output may be expressed as continuous functions of a finite number of its first derivatives and the function is Lipschitz on compact sets. If the system is restricted to a compact subset of the state space and is locally observable, the values of the multiobserver's output are finite sets.

3.1 The construction of multiobservers

The construction of multiobservers for locally observable systems can be divided into the following steps:

(1) The sequence $(L_f^i h)_{i\geq 0}$, is computed and the first N functions

$$y_0 = h, y_1 = L_f h, \dots, y_{N-1} = L_f^{N-1}h,$$

that determine the indistinguishability relation, are chosen.

(2) Higher order derivatives of the output are computed as a continuous function of lower order derivatives, in particular

$$L_f^N h = \varphi_N(y_0, y_1, \dots, y_{N-1}, u, \dots u^{(N-1)}).$$

(3) Let us consider the map $\mathbf{S}\Phi_N : \Omega \times \mathbb{R}^{N-1} \to \mathbb{R}^N \times \mathbb{R}^{N-1}$, $\mathbf{S}\Phi_N(x, \underline{u}_{N-2}) = (h(x), \dots, L_f^{N-1}h(x), \underline{u}_{N-2})$, where $\underline{u}_k = (u, \dots u^{(k)}) \in \mathbb{R}^{k+1}$

(4) x and $\tilde{x}$ – the indistinguishable states (denote: $x \sim \tilde{x}$) are considered and then $\widetilde{\Omega} := \Omega/_\sim$, $\mathbf{S}\widetilde{\Phi}_N : \widetilde{\Omega} \times \mathbb{R}^{N-1} \to \mathbb{R}^N \times \mathbb{R}^{N-1}$,

$\mathbf{S}\widetilde{\Phi}_N([x], \underline{u}_{N-2}) := \mathbf{S}\Phi_N(x, \underline{u}_{N-2})$, $\mathbf{S}\widetilde{\Phi}_N$ is continuous and injective, so there is a continuous map $\mathbf{S}\Psi := \mathbf{S}\widetilde{\Phi}_N^{-1} : \mathbf{S}\widetilde{\Phi}_N(\widetilde{\Omega} \times \mathbb{R}^{N-1}) \to \widetilde{\Omega} \times \mathbb{R}^{N-1}$ and $\mathbf{S}\Psi(y_0, \dots, y_{N-1}, \underline{u}_{N-2}) = \left(\Psi(y_0, \dots, y_{N-1}, \underline{u}_{N-2}), \underline{u}_{N-2}\right)$. Let $\Psi_u(y_0, \dots, y_{N-1}) := \Psi(y_0, \dots, y_{N-1}, \underline{u}_{N-2})$ where $\Psi : \mathbf{S}\widetilde{\Phi}_N(\widetilde{\Omega} \times \mathbb{R}^{N-1}) \to \widetilde{\Omega}$.

(5) Ψ is extended to a continuous multivalued mapping (multifunction) g on entire $\mathbb{R}^N$,

$$g : \mathbb{R}^N \times \mathbb{R}^{N-1} \to \widetilde{\Omega}.$$

(6) A continuous extension φ of the function φ_N is found (φ is Lipschitz, in order to have exponential estimation of the derivatives of the outputs). φ appears in the dynamics of the multiobserver, so its dynamics is only continuous.

(7) The system $\mathcal{S}_{\Sigma,\theta,\Omega}$:

$$\dot{\mathbf{Z}} = (\mathbf{A} - \mathbf{K}_\theta\mathbf{C})\mathbf{Z} + \mathbf{K}_\theta\mathbf{C}y + \mathbf{b}\varphi(\mathbf{Z}, \underline{u}_{N-1})$$
$$\hat{x} = g(\mathbf{Z}, \underline{u}_{N-2})$$

is a continuous multiobserver of Σ, where

$$\mathbf{Z} \in \mathbb{R}^N,\ \mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{N\times N},\ \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}_{N\times 1},\ \Delta_\theta = \begin{bmatrix} \theta & 0 & 0 & \dots & 0 \\ 0 & \theta^2 & 0 & \dots & 0 \\ 0 & 0 & \theta^3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \theta^N \end{bmatrix}_{N\times N},$$

$\theta \gg 1, \mathbf{C} = \begin{bmatrix} 1 & 0\dots 0 \end{bmatrix}_{1\times N}, \mathbf{K}_\theta = \Delta_\theta \mathbf{K}$, $\mathbf{K}$ is such that $(\mathbf{A} - \mathbf{KC})$ is Hurwitz, φ comes from **(6)**, g comes from **(5)**, and u belongs to the set defined as follows

$$U_{N,B} := \left\{u : \mathbb{R}_+ \to \mathbb{R} : u \in \mathcal{C}^\infty \wedge \left|\frac{d^i u}{dt^i}\right| \le B, 0 \le i \le N-1\right\}$$

Examples of such a construction can be found in [3].

The problem appears when the function φ_N is not Lipschitz on some compact subset and in this case a continuous extension of φ_N, which would be Lipschitz, cannot be found. Therefore it can be improved so that to receive a Lipschitz function on each compact set and in this way the construction of multiobservers would be possible to do not only for locally observable systems, for which the function φ_N is Lipschitz, but also for each locally observable system.

3.2 A family of multiobservers

Let Σ be a locally observable system described by the equations (1) and (2) and $\Omega \subset \mathbb{R}^n$ be a compact subset of $\mathbb{R}^n$. Assume that

Assumption: There exists a family of continuous and Lipschitz functions $\tilde{\varphi}$ parametrized by $A \ge 0$ (with a Lipschitz constant L_B dependent on B) such that for all $\underline{\mathbf{y}} \in \mathbb{R}^N$ the following inequality holds $\left|\tilde{\varphi}\left(\underline{\mathbf{y}}, \underline{u}_{N-1}\right) - \varphi_N\left(\underline{\mathbf{y}}, \underline{u}_{N-1}\right)\right| \le A$, where $\sup_{\underline{\mathbf{y}} \in \mathbb{R}^N} \left|\tilde{\varphi}\left(\underline{\mathbf{y}}, \underline{u}_{N-1}\right) - \varphi_N\left(\underline{\mathbf{y}}, \underline{u}_{N-1}\right)\right| = A$, $\underline{u}_{N-1} = (u, \dots, u^{(N-1)})$ and $\frac{A}{L_B}$ goes to zero as $A \to 0$ and $u \in U_{N,B}$.

Let us take one function from above-mentioned family and then consider a candidate for multiobserver system $\mathcal{S}_{\Sigma,\theta,\Omega,A}$ for Σ:

$$\dot{\mathbf{Z}} = (\mathbf{A} - \mathbf{K}_\theta\mathbf{C})\mathbf{Z} + \mathbf{K}_\theta\mathbf{C}y + \mathbf{b}\tilde{\varphi}(\mathbf{Z}, \underline{u}_{N-1})$$
$$\hat{x} = g(\mathbf{Z}, \underline{u}_{N-2})$$

where $\mathbf{Z} \in \mathbb{R}^N$ and $\mathbf{A}, \mathbf{b}, \mathbf{C}, \mathbf{K}_\theta$ are defined above in point **(7)** of the multiobserver's construction, and g is defined in **(5)** of it.

Theorem 3.1. *For each $\mathbf{Z}(0) \in \mathbb{R}^N$, $x(0) \in \Omega$, $u \in U_{N,B}$ and some constants $\alpha \ge 0$, $\mathbf{s} \ge 0$ the solution $\mathbf{Z}(t)$ of $\mathcal{S}_{\Sigma,\theta,\Omega,A}$ satisfies the following inequality $\|\mathbf{Z}(t) - \Phi_N(x(t), u(t), \dots, u^{(N-2)}(t))\| \le \theta^N \left(\frac{\sqrt{\alpha \mathbf{s}\|\epsilon(0)\|}}{\theta} - \frac{\mathbf{s}A}{\theta^N\sqrt{\alpha(\frac{\theta}{2} - \mathbf{s}L_B)}}\right) \cdot e^{-\alpha(\frac{\theta}{2} - \mathbf{s}L_B)t} + \frac{\mathbf{s}A}{\sqrt{\alpha(\frac{\theta}{2} - \mathbf{s}L_B)}}$, where $\Phi_N : \Omega \times \mathbb{R}^{N-1} \to \mathbb{R}^N$ and $\Phi_N(x(t), u, \dots, u^{(N-2)}) := (y_0, y_1, \dots, y_{N-1})$ and $\theta \gg 1$, A and L_B are constants. A and L_B are such constants that the above-mentioned assumption is true. This inequality holds for all t such that $\{x(s) : 0 \le s \le t\} \subset \Omega$ ($x(t)$ is the solution at time t of Σ starting from $x(0)$).*

The proof of theorem 3.1 is similar to the proof in the uncontrolled case that has been given in [6] and therefore will be omitted. We have to add a control u which belongs to $U_{N,B}$. Observe that multiobservers forming the family are not exact as their output approaches the state of the original system with a (small) error. The error goes to zero as we change a parameter in the family of multiobservers.

Let us consider the following example for which the above-mentioned family of (multi)observers is possible to construct.

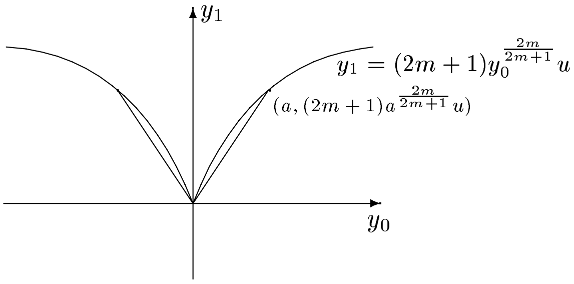
Example 3.2. *Let Σ be the system*

$$\Sigma : \quad \dot{x} = u$$
$$y = x^{2m+1},$$

where $m \in \mathbb{N}, u \ne 0$ and $\exists B : \ |u(t)| \le B$. The system Σ is locally (and globally) observable and the function

$$y_1 = \varphi_1(y_0, u) = (2m+1)y_0^{\frac{2m}{2m+1}}u$$

is continuous but it is not Lipschitz in a neighbourhood of a point $y_0 = 0$.



Because the function φ_1 is not Lipschitz on the compact set $[-a, a]$ $(a > 0)$, it is going to be improved by defining the following function

$$\tilde{\varphi}(y,u) := \begin{cases} \varphi_1(y,u), & \text{if } |y| > a \\ (2m+1)a^{\frac{-1}{2m+1}}|y|u, & \text{if } |y| \le a \end{cases}.$$

The function φ_1 is only a continuous function, but $\tilde{\varphi}$ is continuous and Lipschitz so there is a Lipschitz constant $L_B = B(2m+1)a^{\frac{-1}{2m+1}} \in \mathbb{R}$ such that

$$|\tilde{\varphi}(z_1,u) - \tilde{\varphi}(z_2,u)| \le L_B|z_1 - z_2|,$$

for all $z_1, z_2 \in \mathbb{R}$. Moreover

$$\forall z \in \mathbb{R}: \ \tilde{\varphi}(z,u) \le \varphi_1(z,u).$$

Let us consider the system $\mathcal{S}_{\Sigma,\theta,a}$

$$\dot{z} = -\theta k z + \theta k y + \tilde{\varphi}(z,u) \quad (6)$$

$$\hat{x} = z^{\frac{1}{2m+1}}. \quad (7)$$

If a is small enough, the system $\mathcal{S}_{\Sigma,\theta,a}$ will be a multiobserver of Σ and the solution of the equation (6) will approximate the output y while the equation (7) gives a (single-valued) continuous estimation of $x(t)$. An error, which is made when the solution of (6) approximates the output y, can be estimated by some constant which is small enough.

Set $\epsilon(t) = z(t) - y(t)$. ϵ satisfies the following equation $\frac{d}{dt}\epsilon(t) = -\theta k \epsilon(t) + \tilde{\varphi}(z(t),u) - u(2m+1)\left[y(t)\right]^{\frac{2m}{2m+1}}$. Because

$$0 \le \varphi(y) - \tilde{\varphi}(y) \le u\left(\frac{2m}{2m+1}\right)^{2m} \cdot a^{\frac{2m}{2m+1}}$$

so $\frac{1}{2}\frac{d}{dt}\epsilon^2(t) = -\theta k\epsilon^2(t) + \epsilon(t)\tilde{\varphi}(z(t)) - \epsilon(t)u(2m+1)\left[y(t)\right]^{\frac{2m}{2m+1}} \le (-\theta k + L_B)\epsilon^2(t) + B\left(\frac{2m}{2m+1}\right)^{2m} a^{\frac{2m}{2m+1}}|\epsilon(t)|$,

where $L_B = B \cdot (2m+1) \cdot a^{\frac{-1}{2m+1}}$. *Hence*

$$\frac{1}{2}\frac{d}{dt}\epsilon^2(t) \le \left(-\theta k + B(2m+1)\cdot a^{\frac{-1}{2m+1}}\right)\epsilon^2(t) + +B\left(\frac{2m}{2m+1}\right)^{2m} a^{\frac{2m}{2m+1}}|\epsilon(t)|.$$

Therefore

$$|\epsilon(t)| \le e^{\left(-\theta k + B(2m+1)\cdot a^{\frac{-1}{2m+1}}\right)t}\left(|\epsilon(0)| - \left(\frac{2m}{2m+1}\right)^{2m}\frac{a}{\theta k a^{\frac{1}{2m+1}} - B(2m+1)}\right) + \left(\frac{2m}{2m+1}\right)^{2m}\frac{a}{\theta k a^{\frac{1}{2m+1}} - B(2m+1)}.$$

Hence the error can be estimated as follows $\lim_{t\to+\infty}|\epsilon(t)| \le \left(\frac{2m}{2m+1}\right)^{2m}\frac{a}{\theta k a^{\frac{1}{2m+1}} - B(2m+1)}$ *(because θ can be chosen big enough). If a approaches zero, then $\lim_{t\to+\infty}|\epsilon(t)|$ will approach zero too.*

In this case some family of (multi)observers, which depends on a, is found. The solution of (6) approximates the output y with an error which depends on a, while (7) gives a (single-valued) continuous estimation of $x(t)$.

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