PERFECT OBSERVERS FOR NONLINEAR CONTROL SYSTEMS*

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Abstract. Observers that exactly reconstruct the unknown state of the system has been constructed by Dai. The Dai's concept has been extended by Kaczorek who called this kind of observers perfect observers. The main subject of this paper is to construct such perfect observers for nonlinear observable control systems. There are cosidered systems for which a linear dependence between an input, its derivatives and an output and its derivatives holds. Then, under observability assumption, it is possible to immerse such a system into a singular continuous-time linear control system for which the perfect observer can be designed. Next, the unknown state of original system is reconstructed and thus the perfect observer of considered nonlinear systems is obtained.

Key Words: nonlinear control systems, perfect observers, observability.

1 INTRODUCTION

In 1988 Dai suggested a construction of an observer for a discrete singular system. He has shown that it is possible to construct a singular observer that exactly reconstructs the unknown state of the original system. In [2] Kaczorek has extended the Dai's concept for the singular continuous-time linear systems and he called such observers perfect observers.

Based on the mentioned above papers we want to construct a perfect observer for a nonlinear system. We study here only the simplest case of the nonlinear system with one output and with one input. Using the idea of Jouan and Gauthier [1] we introduce N^{th} dynamical extension of our system. There are two sequences $\left(\mathcal{L}_f^i h\right)_{i\geq 0}$ and $\left(u^{(k)}\right)_{k\geq 0}$ connected with the system, where $u^{(k)} := \frac{d^k u}{dt^k}$ and

$$\mathcal{L}_f\varphi(x, j^{k+1}u) := \frac{\partial\varphi(x,j^k u)}{\partial x} f(x,u) + \sum_{i=0}^{k} \frac{\partial\varphi(x,j^k u)}{\partial u^{(i)}} u^{(i+1)}. \tag{1}$$

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φ depend on $x \in \mathbb{R}^n$ and a finite jet of u: $j^k u := (u, u', u'', \ldots, u^{(k)})$.

We assume that the map that associates the output and its successive derivatives to the state is injective for some control u (then our system is observable) and there is a linear dependence between derivatives of the output and derivatives of the input, namely $\mathcal{L}_f^N h(x, j^{N-1}u) = \alpha_1 h(x) + \alpha_2\mathcal{L}_f h(x,u) + \ldots + \alpha_N \mathcal{L}_f^{N-1} h(x, j^{N-2}u) + \alpha_{N+2}u + \ldots + \alpha_{2N+1}u^{(N-1)}$, where $\alpha_1 \neq 0$ and $\alpha_{N+2} \neq 0$. Then the N^{th} dynamical extension of our system is immersed into a singular continuous-time linear system (it will be shown in Section 3). In this way we obtain the singular linear system for which it is possible to construct the perfect observer. This observer exactly reconstructs the output and its first N derivatives.

Our goal is to reconstruct the unknown state of the original system. If the system is observable then there is a continuous function which depends on output and its derivatives and a perfect observer of original system can be constructed.

In Section 4 there is an example of nonlinear systems for which the perfect observer is constructed.

2 NONLINEAR SYSTEMS AND THEIR N^{th} DYNAMICAL EXTENSIONS

Let us consider a nonlinear system Σ on $\Omega \subset \mathbb{R}^n$:

$$\dot{x} = f(x,u) \quad (2)$$
$$y = h(x) \quad (3)$$

The output $y(t)$ and the input $u(t)$ are assumed to be scalar, for simplicity ($u(\cdot) \in \mathrm{C}^\infty(\mathbb{R}_+)$). The vector field f and the function h are assumed to be smooth on Ω.

Two points x_1 and x_2 in $\mathbb{R}^n$ are *indistinguishable* if

$$h(x(t,x_1,u)) = h(x(t,x_2,u)) \quad (4)$$

for every control u and every $t \geq 0$ for which both sides of the equation (4) are defined. Here $x(t,x_0,u)$ denotes the trajectory of f corresponding to control u and starting at $x(0) = x_0$ and evaluated at time t. Otherwise x_1 and x_2 are called *distinguishable.*

The system Σ is *observable* if any two distinct points are distinguishable.

Let $u_k := u^{(k)}$, $k \geq 1$. Assume that for some $N > 0$ and some $\alpha_1, \dots, \alpha_N, \alpha_{N+2}, \dots, \alpha_{2N+1} \in \mathbb{R}$ the following relationship between functions of the form $\mathcal{L}_f^k h$ and $u^{(k)}$, where $k \geq 0$, holds at each point $x \in \Omega$,

$$\begin{aligned}\mathcal{L}_f^N h(x, j^{N-1}u) &= \alpha_1 h(x) + \alpha_2 \mathcal{L}_f h(x,u) + \cdots \\ &\cdots + \alpha_N \mathcal{L}_f^{N-1} h(x, j^{N-2}u) + \alpha_{N+2} u + \cdots \\ &\cdots + \alpha_{2N+1} u_{N-1},\end{aligned} \quad (5)$$

where $\alpha_1 \neq 0$, $\alpha_{N+2} \neq 0$ and $\mathcal{L}_f$ is defined by (1).

Let us consider the following system

$$\Sigma_N : \begin{cases} \dot{x} = f(x,u) \\ \dot{u} = u_1 \\ \vdots \\ \dot{u}_{N-1} = u_N \\ y = h(x) \end{cases} \quad (6)$$

with new control u_N and state variable $(x, u, u_1, \dots, u_{N-1})$. The system Σ_N will be called N^{th} *dynamical extension of* Σ. Of course, Σ_N satisfies the linear relationship (5) too, but it is the relation between output and its derivatives and new state variable $\underline{u}_{N-1} := (u, u_1, \dots, u_{N-1})$.

3 IMMERSIONS INTO SINGULAR LINEAR CONTINUOUS-TIME SYSTEMS

Let $\tau : \Omega \times \mathbb{R}^N \to \mathbb{R}^{2N+1}$ be the following map

$$\tau(x, \underline{u}_{N-1}) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x,u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, \underline{u}_{N-2}) \\ \mathcal{L}_f^N h(x, \underline{u}_{N-1}) \\ u \\ u_1 \\ \vdots \\ u_{N-1} \end{pmatrix} \quad (7)$$

Let $z_i = \mathcal{L}_f^{i-1} h(x, \underline{u}_{i-2})$ if $i = 1, \dots, N+1$, $z_{N+2} = u$ and $z_i = u_{i-N-2}$ if $i = N+3, \dots, 2N+1$. Then from (5) we obtain an immersion of Σ_N into a singular continuous-time linear system with an input u_N in the following form

$$\Sigma_{\text{singular}} : \begin{cases} E\dot{Z} = AZ + Bu_N \\ y = CZ \end{cases} \quad (8)$$

where $u_N \in \mathbb{R}$ is the input, $Z = (z_1, \dots, z_{2N+1}) \in \mathbb{R}^{2N+1}$ is the state and $E = \begin{pmatrix} \mathrm{I}_{N\times N} & 0 & \mathbf{0}_{N\times N} \\ 0 & 0 & 0 \\ \mathbf{0}_{N\times N} & 0 & \mathrm{I}_{N\times N} \end{pmatrix}$, $\mathrm{I}_{k\times k}$ - identity matrix of dimensions $k \times k$, $\mathbf{0}_{k\times l}$ - zero matrix of dimensions $k \times l$, $A = \begin{pmatrix} \mathbf{0}_{N\times 1} & \mathrm{I}_{N\times N} & \mathbf{0}_{N\times 1} & \mathbf{0}_{N\times i} \\ -\alpha_1 & \mathbf{a^1}_{1\times N} & -\alpha_{N+2} & \mathbf{a^2}_{1\times i} \\ \mathbf{0}_{i\times 1} & \mathbf{0}_{i\times N} & \mathbf{0}_{i\times 1} & \mathrm{I}_{i\times i} \\ 0 & \mathbf{0}_{1\times N} & 0 & \mathbf{0}_{1\times i} \end{pmatrix}$,
$\mathbf{a^2}_{1\times i} = \begin{pmatrix} -\alpha_{N+3} & -\alpha_{N+4} & \dots - \alpha_{2N} & -\alpha_{2N+1} \end{pmatrix}$,
$\mathbf{a^1}_{1\times N} = \begin{pmatrix} -\alpha_2 & -\alpha_3 & \dots & -\alpha_N & 1 \end{pmatrix}$, $i = N-1$,
$B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & c_2 & \dots & c_{2N+1} \end{pmatrix}$,
$c_j = -\frac{\alpha_j}{\alpha_1}$, if $j = 2, \dots, N, N+2, \dots, 2N+1$ and $c_{N+1} = \frac{1}{\alpha_1}$.

4 PERFECT OBSERVERS

Let us consider a linear singular system

$$\begin{aligned} &E\dot{Z} = AZ + Bu, \quad Z(0) = Z_0 \\ &y = CZ, \end{aligned} \quad (9)$$

where $Z(t) \in \mathbb{R}^N$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, $E, A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times m}$ and $C \in \mathbb{R}^{p\times N}$.

Definition 1. ([2]) The system

$$E\dot{\hat{Z}} = A\hat{Z} + Bu + K(C\hat{Z} - y), \; \hat{Z}(0) = \hat{Z}_0, \quad (10)$$

is called *a perfect observer* of the system (9) if $\hat{Z}(t) = Z(t)$, for all $t > 0$ and any initial conditions $Z(0) = Z_0$ of (9) and $\hat{Z}(0) = \hat{Z}_0$ of (10), where $K \in \mathbb{R}^{N\times p}$.

Consider the singular continuous-time linear system with one input of the form (8). The following conditions hold for Σ_{singular}:

(1) $\det E = 0$

(2) $\operatorname{rank}\begin{pmatrix} E \\ C \end{pmatrix} = 2N+1$

(3) $\det(Es - A) \neq 0$, for some $s \in \mathbb{C}$ ($\mathbb{C}$ – the field of complex numbers)

(4) $\operatorname{rank} C = 1$

(5) $\operatorname{rank}\begin{pmatrix} Es - A \\ C \end{pmatrix} = 2N+1$, for all $s \in \mathbb{C}$.

Thus using the techniques of [2], it is possible to construct a perfect observer $\mathcal{O}_{\text{perfect}}$ for the system Σ_{singular} in the following form:

$$E\dot{\hat{Z}} = A\hat{Z} + Bu_N + K(C\hat{Z} - y),\ \hat{Z}(0) = \hat{Z}_0 \quad (11)$$

where K satisfies $\det[Es - (A + KC)] = \beta$, for all $s \in \mathbb{C}$ and β is a nonzero scalar independent of s. Such K exists by [2].

In our case $y \in \mathbb{R}$, so $K = (k_1, \dots k_{2N+1})^{\mathrm{T}} \in \mathbb{R}^{N+1}$ and $\det[Es - (A + KC)] = -(1 + \frac{k_{N+1}}{\alpha_1})s^{2N} + (\alpha_N(1 + \frac{k_{N+1}}{\alpha_1}) + k_1)s^{2N-1} + (\alpha_{N-1}(1 + \frac{k_{N+1}}{\alpha_1}) - \alpha_N k_1 + k_2)s^{2N-2} + (\alpha_{N-2}(1 + \frac{k_{N+1}}{\alpha_1}) - \alpha_{N-1}k_1 - \alpha_N k_2 + k_3)s^{2N-3} + \ldots + (\alpha_3(1 + \frac{k_{N+1}}{\alpha_1}) - \alpha_4 k_1 - \alpha_5 k_2 - \alpha_6 k_3 - \ldots - \alpha_N k_{N-3} + k_{N-2})s^{N+2} + (\alpha_2(1 + \frac{k_{N+1}}{\alpha_1}) - \alpha_3 k_1 - \alpha_4 k_2 - \alpha_5 k_3 - \ldots - \alpha_N k_{N-2} + k_{N-1})s^{N+1} + (\alpha_1 - \alpha_2 k_1 - \alpha_3 k_2 - \alpha_4 k_3 - \ldots - \alpha_N k_{N-1} + k_N)s^N + (\alpha_{N+2}k_{N+2} + \ldots + \alpha_{2N}k_{2N} + \alpha_{2N+1}k_{2N+1})s^{N-1} + (\alpha_{N+2}k_{N+3} + \ldots + \alpha_{2N-1}k_{2N} + \alpha_{2N}k_{2N+1})s^{N-2} + \ldots + (\alpha_{N+2}k_{2N-2} + \alpha_{N+3}k_{2N-1} + \alpha_{N+4}k_{2N} + \alpha_{N+5}k_{2N+1})s^3 + (\alpha_{N+2}k_{2N-1} + \alpha_{N+3}k_{2N} + \alpha_{N+4}k_{2N+1}t)s^2 + (\alpha_{N+2}k_{2N} + \alpha_{N+3}k_{2N+1})s + \alpha_{N+2}k_{2N+1} = \beta \neq 0$ iff $k_i = 0$, if $i = 1, \dots, N-1$, $k_N = -\alpha_N$, $k_{N+1} = -\alpha_{N+1}$, $k_{2N+1} = \frac{\beta}{\alpha_{N+2}}$ and $k_i = \frac{-1}{\alpha_{N+2}} \cdot \left(\sum_{j=0}^{2N-i} \alpha_{N+3+j}k_{i+j+1}\right)$, if $N+2 \le i \le 2N$, where $\alpha_1, \alpha_{N+2}, \beta \neq 0$.

In particular, if $N = 1$ then $K = \begin{pmatrix} -\alpha_1 \\ -\alpha_1 \\ \frac{\beta}{\alpha_3} \end{pmatrix}$.

The perfect observer $\mathcal{O}_{\text{perfect}}$ for the system Σ_{singular} is as follows:

$$\begin{aligned}
\dot{\hat{z}}_1 &= \hat{z}_2 \\
\dot{\hat{z}}_2 &= \hat{z}_3 \\
&\vdots \\
\dot{\hat{z}}_{N-1} &= \hat{z}_N \\
\dot{\hat{z}}_N &= \sum_{i=2}^{N} \alpha_i \hat{z}_i + \sum_{j=N+2}^{2N+1} \alpha_j \hat{z}_j + \alpha_1 y \\
0 &= y - \hat{z}_1 \\
\dot{\hat{z}}_{N+2} &= \hat{z}_{N+3} + \tfrac{k_{N+1}}{\alpha_1}\hat{z}_{N+1} + \sum_{i=2}^{N} \tfrac{\alpha_i}{\alpha_1} k_i \hat{z}_i + \\
&\quad + \sum_{j=N+2}^{2N+1} \tfrac{\alpha_j}{\alpha_1} k_j \hat{z}_j - k_{N+2} y \\
&\vdots \\
\dot{\hat{z}}_{2N} &= \hat{z}_{2N+1} + \tfrac{k_{2N}}{\alpha_1}\hat{z}_{N+1} + \sum_{i=2}^{N} \tfrac{\alpha_i}{\alpha_1} k_i \hat{z}_i + \\
&\quad + \sum_{j=N+2}^{2N+1} \tfrac{\alpha_j}{\alpha_1} k_j \hat{z}_j - k_{2N} y \\
\dot{\hat{z}}_{2N+1} &= u_N + \tfrac{k_{2N+1}}{\alpha_1}\hat{z}_{N+1} + \sum_{i=2}^{N} \tfrac{\alpha_i}{\alpha_1} k_i \hat{z}_i + \\
&\quad + \sum_{j=N+2}^{2N+1} \tfrac{\alpha_j}{\alpha_1} k_j \hat{z}_j - \tfrac{\beta}{\alpha_{N+2}} y
\end{aligned} \quad (12)$$

The system $\mathcal{O}_{\text{perfect}}$ gives the output of Σ and Σ_N and its successive derivatives up to the order N, when $t > 0$, based upon the knowledge of the output $y(t)$ and the input $u(t)$. Now we would like to recover the unknown state $x(t)$ of the original system Σ.

Let Ω be a compact subset of $\mathbb{R}^n$. Let $\Phi_N : \Omega \times \mathbb{R}^{N-1} \to \mathbb{R}^{2N-1}$ be the following map

$$\Phi_N(x, j^{N-2}u) = \begin{pmatrix} h(x) \\ \mathcal{L}_f h(x,u) \\ \vdots \\ \mathcal{L}_f^{N-1} h(x, \underline{u}_{N-2}) \\ (j^{N-2}u)^{\mathrm{T}} \end{pmatrix}.$$

If the map Φ_N associated with the system Σ is injective (so Σ is observable), then there exists its continuous inverse Φ_N^{-1} from $\Phi_N(\Omega \times \mathbb{R}^{N-1})$ onto $\Omega \times \mathbb{R}^{N-1}$. The inverse function Φ_N^{-1} is a continuous map from $\Phi_N(\Omega \times \mathbb{R}^{N-1})$ onto $\Omega \times \mathbb{R}^{N-1}$, so Φ_N^{-1} can be extended to a continuous map Ψ on the entire $\mathbb{R}^{2N-1}$. The system (8) has the property of uniqueness of solutions (because $\det(Es - A) \neq 0$, for some $s \in \mathbb{C}$), so if $(\hat{x}, j^{N-2}u) = \Psi(\hat{z}_1, \dots, \hat{z}_{2N-1}) \in \Omega \times \mathbb{R}^{N-1}$, then $x(t) = \hat{x}(t)$, $t > 0$.

Example 2. Let Σ be the following system on $\Omega \subseteq \mathbb{R} \setminus \{0\}$:

$$\dot{x} = x + \frac{1}{x^2}u$$
$$y = x^3.$$

Let Σ_1 be the 1^{th} dynamical extension of Σ. It is defined as follows

$$\dot{x} = x + \frac{1}{x^2}u$$
$$\dot{u} = u_1$$
$$y = x^3,$$

where u_1 is new input and (x, u) are the state variables of Σ_1. The map $\Phi_1(x) = x^3$ is injective (Σ is observable) and $\mathcal{L}_f h(x, u) = 3x^3 + 3u$ so the perfect observer can be constructed. Using the immersion $\tau(x, u) = (z_1, z_2, z_3)^{\mathrm{T}} = (x^3, 3x^3+3u, u)^{\mathrm{T}}$ we obtain the following singular linear system

$$\Sigma_{\text{singular}} : \begin{cases} \dot{z}_1 = z_2 \\ 0 = -3z_1 - 3z_3 + z_2 \\ \dot{z}_3 = u_1 \\ y = \frac{1}{3}z_2 - z_3 \end{cases},$$

for which there is a perfect observer of the form (12). The map $\Psi(z) = \sqrt[3]{z}$ is a continuous inverse of Φ_1. Then the desired perfect observer has the form

$$\mathcal{O}_{\text{perfect}} : \begin{cases} \dot{\hat{z}}_1 = 3\hat{z}_3 + 3y \\ 0 = y - \hat{z}_1 \\ \dot{\hat{z}}_3 = u_1 + \beta\hat{z}_2 - \frac{\beta}{3}\hat{z}_3 - \frac{\beta}{3}y \\ \hat{x} = \sqrt[3]{\hat{z}_1} \end{cases}.$$

Example 3. Let Σ be the following system on $\Omega \subseteq \mathbb{R} \times (\mathbb{R} \setminus \{0\})$:

$$\dot{x}_1 = x_2^3 + u$$
$$\dot{x}_2 = x_2 + \frac{x_1}{x_2^2}$$
$$y = x_1.$$

Let Σ_2 be the 2^{th} dynamical extension of Σ. It is defined as follows

$$\dot{x}_1 = x_2^3 + u$$
$$\dot{x}_2 = x_2 + \frac{x_1}{x_2^2}$$
$$\dot{u} = u_1$$
$$\dot{u}_1 = u_2$$
$$y = x_1,$$

where u_2 is new input and (x_1, x_2, u, u_1) are the state variables of Σ_2. The map $\Phi_2(x_1, x_2, u) = (x_1, x_2^3+u, u)^{\mathrm{T}}$ is injective (so Σ is observable) and $\mathcal{L}_f^2 h(x_1, x_2, u, u_1) = 3x_2^3+3x_1+u_1 = 3\mathcal{L}_f h(x, u) + 3h(x) - 3u + u_1$, so the perfect observer can be constructed. Using the immersion

$$\tau(x, u, u_1) = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2^3 + u \\ 3x_2^3 + 3x_1 + u_1 \\ u \\ u_1 \end{pmatrix}$$

we obtain the following singular linear system

$$\Sigma_{\text{singular}} : \begin{cases} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ 0 &= z_3 - 3z_1 - 3z_2 + 3z_4 - z_5 \\ \dot{z}_4 &= z_5 \\ \dot{z}_5 &= u_2 \\ y &= -z_2 + \frac{1}{3}z_3 + z_4 - \frac{1}{3}z_5 \end{cases},$$

for which there is a perfect observer of the form (12). The map $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$ defined as follows

$$\Psi(z_1, z_2, z_4) = \begin{pmatrix} z_1 \\ \sqrt[3]{z_2 - z_4} \\ z_4 \end{pmatrix}.$$

is a continuous inverse of Φ_2. Then the desired perfect observer $\mathcal{O}_{\text{perfect}}$ has the form

$$\begin{array}{lcl} \dot{\hat{z}}_1 &=& \hat{z}_2 \\ \dot{\hat{z}}_2 &=& 3\hat{z}_2 - 3\hat{z}_4 + \hat{z}_5 + 3y \\ 0 &=& y - \hat{z}_1 \\ \dot{\hat{z}}_4 &=& (1 + \frac{\beta}{9})\hat{z}_4 - \hat{z}_3 - 3\hat{z}_2 - \frac{\beta}{9}y \\ \dot{\hat{z}}_5 &=& u_2 - \frac{\beta}{9}\hat{z}_3 - 3\hat{z}_2 + \frac{\beta}{9}\hat{z}_4 - \frac{\beta}{9}\hat{z}_5 + \frac{\beta}{3}y \\ \hat{x} &=& \begin{pmatrix} \hat{z}_1 \\ \sqrt[3]{\hat{z}_2 - \hat{z}_4} \end{pmatrix} \end{array}.$$

The following solutions of $\mathcal{O}_{\text{perfect}}$: $\hat{z}_1(t)$, $\hat{z}_2(t)$, $\hat{z}_3(t)$ give the output of Σ and Σ_2 and its successive derivatives up to order 2, when $t > 0$, based upon the knowledge of the output $y(t)$ and the input $u(t)$.

5 REFERENCES

[1] Jouan, P., and Gauthier, J.: Finite Singularities of Nonlinear Systems. Output Stabilization, Observability and Observers. *J. Dynamical and Control Systems*, vol. 2, 1996, pp. 255–288.

[2] Kaczorek, T.: Reduced-order perfect and standard observers for singular continuous-time linear systems. *Machine Intelligence & Robotic Control*, vol. 2, No. 3, 2000,pp. 93–98.

[3] Kaczorek, T.: Full-order perfect observers for continuous-time linear systems. *Bulletin Polish Academy of Sciences, Tech. Sciences*, vol. 49, No. 4, 2001, pp. 549–558.

[4] Kaczorek, T.: Minimal order perfect functional observers for singular linear systems. *Machine Intelligence & Robotic Control*, vol. 4, No. 2, 2002,pp. 71–74.

[5] Xia, X.-H., and Zeitz, M.: On nonlinear continuous observers. *Int. Journal of Control*, vol. 66, 1997, pp. 943–954.