LINEAR CONTROL SYSTEMS ON TIME SCALE: UNIFICATION OF CONTINUOUS AND DISCRETE*

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Abstract. Linear time invariant control systems on arbitrary time scales are studied. The theory unifies discrete-time and continuous-time cases. The basics of delta differential and integral calculus on time scales are presented. The standard results on controllability and observability are extended to systems on arbitrary time scales.

Key words: Time scale, Continuous-time and discrete-time systems, Controllability and observability.

1 Introduction

Control theory consists of two parallel branches: in the first one time is continuous, in the second one – discrete. The statements in both theories are usually similar or even identical. This is true for linear as well for nonlinear systems. The language of time scales, created some 25 years ago by Stefan Hilger [2] in his Ph.D. thesis, seems to be an ideal tool to unify the two theories. One of the main concepts is the delta derivative, which is a generalization of ordinary (time) derivative. If the time scale is the real line, we get ordinary derivative. In the case of integer numbers, delta derivative of a function is the difference of its values at subsequent points. Thus differential equations as well difference equations are naturally included in the theory. But one can also consider mixed cases when time is partly discrete and partly continuous (see [1] for examples in engineering). We develop here the basics of linear control theory on time scales. We assume that matrices that appear in the description of systems have constant coefficients. In particular, we present criteria of controllability and observability for systems on time scales. Formally, they are extensions of well known facts in both discrete-time and continuous-time theories.

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2 Calculus on time scales

We give here a short introduction to differential calculus on time scales. This is a generalization of the standard differential calculus, on one hand, and the calculus of finite differences, on the other hand. Then we describe the inverse operation — integration. This will allow to solve differential equations on time scales. More material on this subject can be found in [1].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. For $t \in \mathbb{T}$ we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

Finally we define the set $\mathbb{T}^k$ in the following way: $\mathbb{T}^k := \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ if $\sup \mathbb{T} < \infty$ and $\mathbb{T}^k := \mathbb{T}$ if $\sup \mathbb{T} = \infty$.

Example 2.1.

- If $\mathbb{T} = \mathbb{R}$ then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
- If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.

Definition 2.2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. *Delta derivative* of f at t, denoted by $f^{\triangle}(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^{\triangle}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

It can be noticed that in the general case forward jump operator σ is not delta differentiable in points t_0 such that $t_0 < \sigma(t_0)$.

Remark 2.3.

- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

  $$f^{\triangle}(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

  i.e. iff f is differentiable in the ordinary sense at t.
- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with

  $$f^{\triangle}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)} =$$

  $$f(t+1) - f(t) = \triangle f(t)$$

  where $\triangle$ is the usual forward difference operator defined by the last equation above.

Example 2.4. The delta derivative of t^2 is $t + \sigma(t)$. This means that the second delta derivative of t^2 may not exists.
The delta-derivative of $\frac{1}{t}$ is $\frac{-1}{t\sigma(t)}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points at $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. It can be shown that
f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated
and that σ is rd-continuous.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are n times delta differentiable and whose all derivatives are rd-continuous will be called of the C^n *rd-class*.

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D, provided $D \subset \mathbb{T}^k$, $D \setminus \mathbb{T}^k$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. It can be proved that if f is regulated then there exists a function F that is pre-differentiable with region of differentiation D such that

$$F^{\triangle}(t) = f(t)$$

for all $t \in D$. Any such function is called pre-antiderivative of f. Then *indefinite integral* of f is defined by

$$\int f(t)\triangle t = F(t) + C$$

where C is an arbitrary constant. *Cauchy integral* is

$$\int_r^s f(t)\triangle t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}^k$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

Remark 2.5. It can be shown that every rd-continuous function has an antiderivative. Moreover, if $f(t) \geq 0$ for all $a \leq t < b$ and $\int_a^b f(\tau)\triangle\tau = 0$ then $f \equiv 0$.

Example 2.6.

- If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau)\triangle\tau = \int_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral.
- If $\mathbb{T} = \mathbb{Z}$, then $\int_a^b f(\tau)\triangle\tau = \sum_{t=a}^{b-1} f(t)$ for $a < b$.
- If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau)\triangle\tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

Remark 2.7. An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula of an antiderivative of t: antiderivative of $\frac{t^2}{2}$ is $\frac{t+\sigma(t)}{2} = t + \frac{\mu(t)}{2}$.

Example 2.8. If $\mathbb{T} = \mathbb{Z}$ and $a \neq 1$, then $\int a^t \triangle t = \frac{a^t}{a-1} + C$, since $(\frac{a^t}{a-1})^{\triangle} = \frac{a^{t+1} - a^t}{a-1} = a^t$.

3 Systems of differential equations

We shall consider here systems of linear differential equations with constant coefficients, defined on time scales.

An $n \times n$ matrix A is called *regressive* with respect to $\mathbb{T}$ provided $I + \mu(t)A$ is invertible for all $t \in \mathbb{T}^k$ (I denotes the identity matrix). The system of delta differential equations $x^{\triangle} = Ax$ is called *regressive* provided A is regressive.

Remark 3.1. We have the following properties:

- If $\mathbb{T} = \mathbb{R}$, then any matrix A satisfies the regressivity condition.
- If $\mathbb{T} = \mathbb{Z}$, then A is regressive if and only if the matrix $I + A$ is invertible, which holds if and only if -1 is not an eigenvalue of A. In this case the equation $x^{\triangle} = Ax$ is equivalent to $x(t+1) = (I + A)x(t)$. Regressivity means that the last equation can be solved backwards.

Theorem 3.2. [1] *Let A be a regressive $n \times n$ matrix. Then the initial value problem*

$$x^{\triangle} = Ax, \quad x(t_0) = x_0$$

has a unique solution x defined on $\mathbb{T}$.

Let $t_0 \in \mathbb{T}$ and let A be regressive. The unique matrix-valued solution of the initial value problem $X^{\triangle} = AX$, $X(t_0) = I$, is called the *matrix exponential function of* A (at t_0). Its value at $t \in \mathbb{T}$ will be denoted by $e_A(t, t_0)$

Remark 3.3. Let A be an $n \times n$ matrix.

- If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$.
- If $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then $e_A(t, t_0) = (I + A)^{(t-t_0)}$.
- If $\mathbb{T} = 2^{\mathbb{N}_0}$ and A is a regressive constant matrix, then $e_A(t, 1) = \prod\limits_{s \in \mathbb{T} \cap (0,t)} (I + sA)$ ordered as: $(I = \rho(t)A) \ldots (I + 2A)(I + A)$.

It can be proved that

1. $e_0(t, s) = I$ and $e_A(t, t) = I$ for every $t, s \in \mathbb{T}$;
2. $e_A(\sigma(t), s) = (I + \mu(t)A)e_A(t, s)$;
3. $e_A(t, s) = e_A^{-1}(s, t)$;
4. $e_A(t, s)e_A(s, r) = e_A(t, r)$.

Theorem 3.4. (Putzer Algorithm) *Let A be a regressive $n \times n$ matrix and $t_0 \in \mathbb{T}$. If $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A, then*

$$e_A(t, t_0) = \sum_{i=0}^{n-1} r_{i+1}(t) P_i$$

where $r(t) := (r_1(t), \ldots, r_n(t))^T$ is the solution of the initial value problem

$$r^{\triangle} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 1 & \lambda_2 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & 1 & \lambda_n \end{pmatrix} r, \quad r(t_0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

and the matrices $P_0, P_1, \ldots, P_n$ are recursively defined by $P_0 = I$ and

$$P_{k+1} = (A - \lambda_{k+1} I)P_k, \quad \text{for } k = 0, 1, \ldots, n-1$$

Let us consider the nonhomogenous equation

$$x^{\triangle}(t) = Ax(t) + f(t), \quad x(t_0) = x_0 \tag{1}$$

where $f : \mathbb{T} \to \mathbb{R}^n$ is a vector-valued rd-continuous function, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$.

Theorem 3.5. *Let A be a regressive $n \times n$ matrix. Then the initial value problem (1) has a unique solution, given by*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau)) f(\tau) \triangle\tau \tag{2}$$

4 Linear time-invariant control systems

In this section we extend standard results on controllability and observability (see e.g. [3, 4]) to linear systems on time scales. They correspond directly to similar results for the continuous-time case. However, the discrete time case is more involved, as the delta derivative in this case gives the difference operator and not the time shift, which is usually used.

Let us consider a linear time-invariant control system with output

$$x^{\triangle}(t) = Ax(t) + Bu(t) \tag{3}$$
$$y(t) = Cx(t) + Du(t) \tag{4}$$

Suppose that $t \in \mathbb{T}$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and A, B, C, D are constant matrices of dimensions respectively $n \times n$, $n \times m$, $p \times n$ and $p \times m$. Moreover, we assume that A is regressive and controls u are piecewise rd-continuous.

Recall that

- system (3), (4) is *controllable* if for any two states $x_0, x_f \in \mathbb{R}^n$ there exist $t_0, t_f \in \mathbb{T}$, $t_f > t_0$, and control $u(t)$, $t \in [t_0, t_f] \cap \mathbb{T}$ such that for $x_0 = x(t_0)$ one has $x(t_f) = x_f$.
- two states $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* if for every control u and for every time $t \in \mathrm{dom} u = [t_0, t_u]$ the value of the output $y(t)$ corresponding to u is the same for both initial conditions $x(t_0) = x_1$ and $x(t_0) = x_2$. System (3), (4) is *observable* if any two indistinguishable states are equal.

We shall assume in this section that the time scale $\mathbb{T}$ consists of at least n elements.

Theorem 4.1. *The system (3) is controllable if and only if*

$$\mathrm{rank}(B, AB, \ldots, A^{n-1}B) = n. \tag{5}$$

Idea of the proof: First we show that controllability is equivalent to the condition

$$\mathrm{rank}(P_0B, P_1B, \ldots, P_{n-1}B) = n, \tag{6}$$

where the matrices $P_0, \ldots,$ P_{n-1} are given in the Putzer algorithm. The proof of necessity goes as in the classical continuous-time case. To prove sufficiency we use the Putzer algorithm. To get condition (6) we need employ the differential equations that define the functions r_i, $i = 1, \ldots, n$. Then we show that conditions (5) and (6) are equivalent.

Similarly we can show the following

Theorem 4.2. *The system (3) is observable if and only if*

$$\mathrm{rank}\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix} = n. \tag{7}$$

5 Conclusions

We presented here extension of controllability and observability results to systems on arbitrary time scales. We studied only linear time-invariant systems, so the whole area of nonlinear control is waiting for a lift to arbitrary time scales. Moreover, we have assumed that the matrix of coefficients of the linear system is regressive, which for discrete-time systems implies existence of backward solutions. Definitely this is a restrictive assumption in control theory and one may try to relax it. This would, however, require developing a new theory of linear differential equations on time scales.

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