Local observability of systems on $\mathbb{R}^\infty$

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Abstract

Nonlinear systems on $\mathbb{R}^\infty$, with output, are investigated. They are described by functions depending on finite number of variables as in the case of the infinite dynamic extensions of finite–dimensional control systems. Local observability of such systems is introduced. It may be described as the property that locally one can recover the value of each state variable by using finitely many functions from the observation algebra. A criterion of local observability is given. It is expressed with the aid of real radicals.

1 Motivation

We study here a particular type of infinite-dimensional nonlinear analytic systems with output. They are defined on $\mathbb{R}^\infty$ and described by analytic functions such that each function depends on a finite number of variables, but these variables are, in general, different for each function. This finiteness feature allows us to use the methods from the finite–dimensional theory to study local observabilty.

A motivation for studying such systems is the concept of infinite dynamic extension of a control system. Such an extension appeared in work of Fliess [3], Pomet [9], Jakubczyk [6] and others. Let us consider a control system with output

$$(\Sigma): \begin{array}{l} \dot{x} = f(x,u) \\ y = h(x,u), \end{array} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}$, $u(t) \in \mathbb{R}$. We assume that controls u are smooth functions of time t. We enlarge the state variables adding u and all its derivatives. Since they are related to each other we obtain the infinite dynamic extension of system Σ, denoted by Σ_∞ and described by the following system of differential equations and the output equation:

$$(\Sigma_\infty): \begin{array}{l} \dot{x} = f(x,u) \\ \dot{u} = u_1 \\ \dot{u}_1 = u_2 \\ \vdots \\ \dot{u}_k = u_{k+1} \\ \vdots \\ y = h(x,u). \end{array} \tag{2}$$

For simplicity we use for Σ_∞ the following notation. Let $\mathcal{U} = (u_0, u_1, u_2, \ldots)$ be the infinite sequence of control $u = u_0$ and its derivatives and

$$\mathcal{X} = (x, u_0, u_1, \ldots) = (x, \mathcal{U}).$$

Then Σ_∞ is defined by a vector field on $\mathbb{R}^\infty$ (i.e. a differential operator) of the form

$$F(\mathcal{X}) = \sum_{i=1}^{n} f_i(x,u_0)\frac{\partial}{\partial x_i} + \sum_{j=0}^{\infty} u_{j+1}\frac{\partial}{\partial u_j} \tag{3}$$

and output function $h_\infty : \mathbb{R}^\infty \longrightarrow \mathbb{R}$, where $h_\infty(\mathcal{X}) = h(x,u_0) = (h \circ \pi^0)(\mathcal{X})$ and π^k is the projection:

$$\pi^k(\mathcal{X}) = (x, u_0, \ldots, u_k),\ k \in \mathbb{N} \cup \{0\}.$$

By the Lie derivative of a function $\varphi : \mathbb{R}^\infty \to \mathbb{R}$ with respect to the vector field F we mean the function:

$$(L_F\varphi)(\mathcal{X}) = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(\mathcal{X}) f_i(\Pi^0(\mathcal{X})) + \sum_{j=0}^{\infty} u_{j+1}\frac{\partial \varphi}{\partial u_j}(\mathcal{X}),$$

where φ depends on finite number of variables. Observe that

$$(L_F h_\infty)(\mathcal{X}) = \sum_{i=1}^{n} \frac{\partial h}{\partial x_i}(x,u_0) f_i(x,u_0) + \frac{\partial h}{\partial u_0}(x,u_0)u_1$$

depends on $\pi^1(\mathcal{X}) = (x,u_0,u_1)$ and $(L_F^k h_\infty)(\mathcal{X})$ depends on $\pi^k(\mathcal{X})$. Let

$$\mathcal{H}(\Sigma_\infty) = \mathbb{R}[L_F^k h_\infty,\ k \in \mathbb{N} \cup \{0\}]$$

be the observation algebra of Σ_∞. Then each function in $\mathcal{H}(\Sigma_\infty)$ depends on finite number of variables. This is an important feature of our study.

2 Local observability of finite–dimensional systems

We recall here basic facts from finite–dimensional theory. The details can be found in [1].

Let us consider a control system Σ of the form (1), where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^r$, $u(t) \in \Omega$ — an arbitrary set. By $\gamma(t, x_0, u)$ we denote the solution of $\dot{x} = f(x, u)$ corresponding to the initial condition $\gamma(0, x_0, u) = x_0$ and control u, and evaluated at time t.

We say that $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* (with respect to Σ) if

$$h(\gamma(t, x_1, u), u(t)) = h(\gamma(t, x_2, u), u(t))$$

for every control u and every $t \geq 0$ for which both sides are defined. Otherwise x_1 and x_2 are *distinguishable.*

We say that Σ is *locally observable at* x_0 (LO(x_0)) if there is a neighborhood U of x_0 such that for every $x \in U$, x and x_0 are distinguishable.

By the observation algebra of system Σ, denoted by $\mathcal{H}(\Sigma)$, we mean the smallest algebra over $\mathbb{R}$ of real analytic functions on $\mathbb{R}^n$ containing the components of the map h and closed under the Lie derivatives with respect to the vector fields of Σ (corresponding to constant controls).

Proposition 2.1. *Points x_1 and x_2 are indistinguishable iff for every $\varphi \in \mathcal{H}(\Sigma)$, $\varphi(x_1) = \varphi(x_2)$.*

Remark 2.1. In the view of Proposition 2.1 local observability at x_0 may be interpreted as the property that the level set of $\mathcal{H}(\Sigma)$ passing through x_0 consists locally (i.e. in a neighborhood of x_0) of one point (of course x_0).

Because we need only the local information about functions and sets, we use the language of germs (see [4] for definitions). In particular, local observability at x_0 may be restated as the property that the germ at x_0 of the level set of $\mathcal{H}(\Sigma)$ passing through x_0 is equal x_0.

By $\mathcal{O}_x^n$ we denote the algebra of germs of real analytic functions at x, where $x \in \mathbb{R}^n$, and by m_x^n the maximal ideal of $\mathcal{O}_x^n$, consisting of all germs in $\mathcal{O}_x^n$ that vanish at x. This is the only maximal ideal of $\mathcal{O}_x^n$.

Let I be an ideal of $\mathcal{O}_x^n$. Then $Z(I)$ denotes the zero set–germ of I (at x). If G_x is a set–germ at point $x \in \mathbb{R}^n$, then $J(G_x)$ denotes the ideal of $\mathcal{O}_x^n$ of germs (at x) of real analytic functions that vanish on G_x.

Now let I be an ideal of any ring P. Then the *real radical of* I, denoted by $\sqrt[\mathbb{R}]{I}$, is the set of all elements $a \in P$ for which there is $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $b_1, \ldots, b_k \in P$ such that

$$a^{2m} + b_1^2 + \cdots + b_k^2 \in I.$$

One can show that $\sqrt[\mathbb{R}]{I}$ is again an ideal of P and $I \subset \sqrt[\mathbb{R}]{I}$.

The following theorem, proved by J.-J. Risler, is the key fact in characterization of local observability.

Theorem 2.1. ([10]) *Let $x \in \mathbb{R}^n$, I an ideal of $\mathcal{O}_x^n$. Then $J(Z(I)) = \sqrt[\mathbb{R}]{I}$.* □

Let I_{x_0} be the ideal of $\mathcal{O}_{x_0}^n$ generated by the germs at x_0 of those functions from $\mathcal{H}(\Sigma)$ that vanish at x_0.

Theorem 2.2. ([1]) *System Σ is LO(x_0) iff $\sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}^n$.* □

Corollary 2.1. *If $I_{x_0} = m_{x_0}^n$ then Σ is LO(x_0).* □

Remark 2.2. The sufficient condition that appeared in Corollary 2.1 is equivalent to the well-known Hermann-Krener rank condition [5]. Usually it is easier to check this condition than the necessary and sufficient condition of Theorem 2.2.

3 Infinite–dimensional systems

Let us identify $\mathbb{R}^\infty$ with $\mathbb{R}^\mathbb{N} = \{x : \mathbb{N} \to \mathbb{R}\}$. For a subset $S \subset \mathbb{N}$, finite or infinite, and $x \in \mathbb{R}^\mathbb{N}$, $x_{|S} : S \longrightarrow \mathbb{R}$ is the restriction of x to S. Let $S \subset S' \subset \mathbb{N}$. By $\Pi_S^{S'}$ we denote the projection

$$\Pi_S^{S'} : \mathbb{R}^{S'} \longrightarrow \mathbb{R}^S : x_{|S'} \longrightarrow x_{|S}.$$

For $S' = \mathbb{N}$ we write $\Pi_S^\mathbb{N} = \Pi_S$.

We say that a function $\varphi : \mathbb{R}^\mathbb{N} \longrightarrow \mathbb{R}$ *depends on finite number of variables* if there exists a finite nonempty subset $S \subset \mathbb{N}$ and a function $\tilde{\varphi} : \mathbb{R}^S \longrightarrow \mathbb{R}$ such that $\varphi = \tilde{\varphi} \circ \Pi_S$. We say that $\varphi : \mathbb{R}^\mathbb{N} \longrightarrow \mathbb{R}$ is *analytic* if φ depends on finite number of variables and $\tilde{\varphi}$ is analytic.

Now we define a system on $\mathbb{R}^\mathbb{N}$, denoted by $\Sigma_\mathbb{N}$, as a (formal) vector field $f = \sum_{i \in \mathbb{N}} f_i \frac{\partial}{\partial x_i}$ and the output function $h : \mathbb{R}^\mathbb{N} \longrightarrow \mathbb{R}^A$, A an arbitrary set, where $f_i, i \in \mathbb{N}$ and $h_j, j \in A$ are analytic on $\mathbb{R}^\mathbb{N}$. The systems we consider here are without control, but all the definitions and results may be formulated for control systems with output.

Let $x_0 \in \mathbb{R}^\mathbb{N}$. We say that the mapping $[0, a) \ni t \longrightarrow \gamma(t, x_0) \in \mathbb{R}^\mathbb{N}$ is a (forward) solution of system $\Sigma_\mathbb{N}$, corresponding to the initial condition $\gamma(0, x_0) = x_0$, if for all $i \in \mathbb{N}$ and $t \in [0, a)$

$$\frac{d}{dt}(\Pi_{\{i\}}\gamma)(t, x_0) = f_i(\gamma(t, x_0)).$$

We shall assume existence and uniqueness of trajectories.

We say that $x_1, x_2 \in \mathbb{R}^{\mathbb{N}}$ are *indistinguishable* if

$$\forall (t \geq 0)\ \forall (i \in A)\ h_i(\gamma(t, x_1)) = h_i(\gamma(t, x_2)). \qquad (4)$$

Let $\tilde{h}_i$ be such that $h_i = \tilde{h}_i \circ \Pi_{S_i}$. Then the condition (4) can be written in the form

$$\forall (t \geq 0) \forall (i \in S)\ \tilde{h}_i(\Pi_{S_i}(\gamma(t, x_1))) = \tilde{h}_i(\Pi_{S_i}(\gamma(t, x_2))).$$

By the Lie derivatives of an analytic function $\varphi : \mathbb{R}^{\mathbb{N}} \longrightarrow \mathbb{R}$, with respect to the vector field f, we mean

$$(L_f \varphi)(x) = \sum_{i \in \mathbb{N}} f_i(x) \frac{\partial \varphi}{\partial x_i}(x).$$

Observe that this new function depends again on finite number of variables and is analytic. By the *observation algebra* of the system $\Sigma_{\mathbb{N}}$, denoted by $\mathcal{H}(\Sigma_{\mathbb{N}})$, we mean the smallest subalgebra of the algebra of real analytic functions on $\mathbb{R}^{\mathbb{N}}$ containing all components of h and closed under the Lie derivatives with respect to f.

Proposition 3.1. *Points $x_1, x_2 \in \mathbb{R}^{\mathbb{N}}$ are indistiguishable iff $\varphi(x_1) = \varphi(x_2)$ for every $\varphi \in \mathcal{H}(\Sigma_{\mathbb{N}})$.* □

There are potentially many ways of defining local observability for systems on $\mathbb{R}^{\mathbb{N}}$. It appears, however, that simple copying of the definition given for the finite-dimensional case does not work. The main obstacle is that there is no appropriate topology on $\mathbb{R}^{\mathbb{N}}$ (which is responsible for a meaning of "neighborhood") that could satisfy natural requirements concerning germs of sets (see [8] for more details). We take then a different approach, exploiting the fact that the functions we use depend on finite number of variables.

We shall first define local observability of an arbitrary family of real functions on $\mathbb{R}^{\mathbb{N}}$ and then define local observability of $\Sigma_{\mathbb{N}}$ as local observability of its observation algebra (using Proposition 3.1). Let $\mathcal{F}$ be any family of real analytic functions on $\mathbb{R}^{\mathbb{N}}$ and $x_0 \in \mathbb{R}^{\mathbb{N}}$. The idea of local observability of $\mathcal{F}$ at x_0 is as follows. We consider systems of equations $\varphi_1(x) = \varphi_1(x_0)$, $\ldots$, $\varphi_k(x) = \varphi_k(x_0)$, where $\varphi_i \in \mathcal{F}$ and $k \in \mathbb{N}$. We want to deduce from such a system that locally around $x_{0i} = (x_0)_{|\{i\}}$, $x_i = x_{0i}$ is the only value of variable x_i that satisfies all the systems. The important fact, consistent with the general approach, is that we use only finite number of functions from $\mathcal{F}$ to achieve this. Unfortunately, we cannot describe this property as in the finite-dimensional case using neighborhood in some topology on $\mathbb{R}^{\mathbb{N}}$.

Let $\varphi_1, \ldots, \varphi_k \in \mathcal{F}$ depend on $x_i, i \in S$. Then $W(\varphi_1, \ldots, \varphi_k)$ denotes the germ at $x_{0|S}$ in $\mathbb{R}^S$ of the level set of $\varphi_1, \ldots, \varphi_k$. We say that $\mathcal{F}$ is *locally observable* at $x_0 \in \mathbb{R}^{\mathbb{N}}$ if for every finite $S \subset \mathbb{N}$ there are $\varphi_1, \ldots, \varphi_k \in \mathcal{F}$ depending on $x_i, i \in S'$ such that $S \subset S'$ and $W(\varphi_1, \ldots, \varphi_k) \subset (\{x_{0|S}\} \times \mathbb{R}^{S' \setminus S})_{x_{0|S'}}$. The last relation is an inclusion of germs in $\mathbb{R}^{S'}$. It means that equations $\varphi_1(x_{|S'}) = \varphi_1(x_{0|S'}), \ldots, \varphi_k(x_{|S'}) = \varphi_k(x_{0|S'})$ imply that locally around $x_{0|S'}$, $x_i = x_{0i}$ for $i \in S$. Though this property at the moment cannot be extended to the remaining variables x_i for $i \in S' \setminus S$, but taking more functions from $\mathcal{F}$ will give this.

Example 3.1. Let $\mathcal{F}$ consists of functions φ_i, $i \in \mathbb{N}$, where $\varphi_i(x) = x_{2i-1}^2 + (x_{2i} - x_{2i+1})^2$ and $x_0 = 0$. Then $\varphi_1(x) = 0$ implies $x_1 = 0$. To get that $x_2 = 0$ and $x_3 = 0$ we must use also $\varphi_2(x) = 0$ which involves x_4 and x_5. Thus considering any finite subset of $\mathcal{F}$ we cannot deduce that all the variables that appear there are equal to 0. In other words, the germ at 0 of the level set of any finite subset of $\mathcal{F}$ never reduces to 0. Nevertheless, $\mathcal{F}$ is locally observable since we get that for any $i \in \mathbb{N}$, $x_i = 0$, using finitely many functions from $\mathcal{F}$.

Now we can finally define that $\Sigma_{\mathbb{N}}$ is *locally observable* at x_0 if $\mathcal{H}(\Sigma_{\mathbb{N}})$ is locally observable at x_0.

To characterize local observability let us consider the algebra of germs of analytic functions at $x_0 \in \mathbb{R}^{\mathbb{N}}$, denoted by $\mathcal{O}_{x_0}^{\mathbb{N}}$. Every such germ is identified with a germ of an analytic function on some $\mathbb{R}^k$. To add or multiply two germs we identify them with germs of analytic functions on some common $\mathbb{R}^k$. By $m_{x_0}^{\mathbb{N}}$ we will denote the ideal of $\mathcal{O}_{x_0}^{\mathbb{N}}$ generated by $x_i - x_{0i}$, $i \in \mathbb{N}$. It is the only maximal ideal of $\mathcal{O}_{x_0}^{\mathbb{N}}$.

Let I_{x_0} be the ideal of $\mathcal{O}_{x_0}^{\mathbb{N}}$ generated by the germs of functions $\varphi - \varphi(x_0)$, $\varphi \in \mathcal{H}(\Sigma_{\mathbb{N}})$. Then the main result may be stated as follows:

Theorem 3.1. *System $\Sigma_{\mathbb{N}}$ is locally observable at x_0 iff $\sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}^{\mathbb{N}}$.* □

This result is a natural extension of the corresponding characterization for finite-dimensional analytic systems.

Example 3.2. Let us consider the system

$$\begin{aligned}
\dot{x}_1 &= x_1 x_2 \\
\dot{x}_2 &= -x_2^2 + x_3 \\
\dot{x}_3 &= -x_2 x_3 + x_4 \\
&\vdots \\
\dot{x}_n &= -x_2 x_n + x_{n+1} \\
&\vdots \\
y &= x_1.
\end{aligned}$$

Then the observation algebra of the system is generated by the functions: x_1, $x_1 x_2$, $x_1 x_3$, $\ldots$, $x_1 x_n$, $\ldots$. Let

$x_0 = 0$. Then we obtain that $x_1 = 0$, but other variables need not be zero, so the system is not locally observable at 0. The ideal I_0 is generated by x_1 and $x_1 x_n$ for $n > 1$ and is equal to its real radical, so $\sqrt[\mathbb{R}]{I_0} \neq m_0^{\mathbb{N}}$.

4 Conclusion

We presented a new definition of local observability of nonlinear systems with an infinite–dimensional state space. It naturally extends its finite–dimensional counterpart. Local observability was characterized by maximality of the real radical of the ideal associated with the system. This again extends the criterion proved earlier for the finite–dimensional case.

Local observability does not have to be stable. A necessary and sufficient condition for stable local observability of a finite-dimensional system was given in [2, 7]. A nice problem is to transfer this characterization to the infinite–dimensional case.

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