Dualities for linear control differential systems with infinite matrices

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Abstract: Infinite–dimensional linear dynamic systems described by infinite matrices are studied. Approximate controllability for systems with lower-diagonal matrices is investigated, whereas observability is studied for systems with row-finite and upper-diagonal matrices. Different necessary or sufficient conditions of approximate controllability and observability of such systems are given. They are used to show dualities between these properties. The theorems on dualities extend the results known for finite-dimensional systems.
Key–words: Infinite linear differential system, row–finite infinite matrix, approximate controllability, observability.

1 Introduction

Infinite-dimensional dynamical systems appear usually either as systems described by partial differential equations or abstract systems defined on Banach spaces. The dynamics of the system is then described by the equation $\dot{x} = Ax$, where A is a linear operator whose domain is usually a dense subspace of a Banach space X, [5, 13, 21, 24]. When the space X is a space of infinite sequences, A may be identified with an infinite matrix and we obtain an infinite system of linear ordinary differential equations. Such systems appear in many applications, [6, 26, 27]. As the state space is now some Banach space of real sequences, Banach space theory can be applied, [6, 19]. In control theory such infinite systems may appear if one considers the infinite extension of a finite-dimensional system, [8, 10, 20]. The new variables are the derivatives of the control. In this case the natural state space is the space of all real sequences. It is a Fréchet space, but not a Banach space, so the theory of systems on Banach spaces cannot be used.

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Another interesting example of an infinite differential system is the system obtained by the Carleman linearization of a nonlinear differential system evolving on a finite-dimensional space, [14].

We study here linear systems that are described by infinite matrices, [4, 25]. We restrict our studies to lower-diagonal and upper-diagonal matrices or matrices called row-finite and column-finite.

Let $\mathbb{R}^{\mathbb{N}}$ denote the linear space of all infinite sequences of real numbers represented by infinite columns $x = (x_1, \ldots, x_i, \ldots)^T, x_i \in \mathbb{R}, i \in \mathbb{N}$. Then by $\mathbb{R}^{(\mathbb{N})}$ we denote its linear subspace, the space of infinite sequences with finitely many nonzero elements. Accordingly to the class of matrices of a system we consider as a state space the space $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{R}^{(\mathbb{N})}$.

Let us consider the linear infinite-dimensional initial value problem

$$\dot{x}(t) = Ax(t) + bu(t), \tag{1.1a}$$

$$x(0) = x^0 \in X \subseteq \mathbb{R}^{\mathbb{N}}. \tag{1.1b}$$

where A is an infinite matrix from $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and $b \in \mathbb{R}^{\mathbb{N}}$. The functions $u(\cdot)$ are assumed to be real-valued, locally Lebesgue-integrable functions.

As for some infinite matrices the initial value problem (1.1) may have infinitely many smooth solutions, we admit formal solutions of (1.1a) given by sequences of formal power series, [22]. This is enough in observability problems and allows to have existence and uniqueness of solutions. To study controllability problems we need more regular solutions. Now the formal solutions must be convergent and thus – analytic. This requires however a special structure of the matrix A, like lower- or upper-diagonality, and corresponding restrictions on initial values of solutions.

One of the main problems studied in this paper is controllability of systems with one-dimensional control, described by lower-diagonal matrices. As in this situation the operator through which the control acts on the system is compact, we have a similar situation like in controllability for abstract differential systems on Banach spaces, (see for instance [24]). We show that for such systems exact controllability is never possible and we study a weaker property of approximate controllability which means that the attainable set from zero is dense in $\mathbb{R}^{\mathbb{N}}$.

We recall the results on observability for row-finite systems obtained in [3]. Next we study observability of systems described by upper-diagonal matrices. We formulate and prove necessary and sufficient condition of observability of such systems defined on $\mathbb{R}^{\mathbb{N}}$ or $\mathbb{R}^{(\mathbb{N})}$.

We present two theorems on duality. The first states the equivalence between properties of approximate controllability for systems with lower-diagonal matrices on $\mathbb{R}^{\mathbb{N}}$ and observability for upper-diagonal systems on $\mathbb{R}^{(\mathbb{N})}$. The second shows the connections between observability of systems with row-finite upper-diagonal matrices on the whole space $\mathbb{R}^{\mathbb{N}}$ and approximate controllability of systems with lower-diagonal column-finite matrices.

2 Preliminaries

In this article an important fact is that the space of all real infinite sequences is not a Banach space. The space $\mathbb{R}^{\mathbb{N}}$ is a Fréchet space, i.e. a complete metrizable locally convex (topological vector) space. As the topology in $\mathbb{R}^{\mathbb{N}}$ we consider the product topology. There are different ways of defining a metric in $\mathbb{R}^{\mathbb{N}}$. We can use the formula: $\rho(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{|x_k - y_k|}{1+|x_k-y_k|}$ for $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ in $\mathbb{R}^{\mathbb{N}}$. A sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots)$ converges to the element $y = (y_1, y_2, \ldots)$ iff $\lim\limits_{n\to\infty} x_k^{(n)} = y_k$ for each $k \in \mathbb{N}$. It means that the convergence of sequences of points in $\mathbb{R}^{\mathbb{N}}$ is coordinatewise, [11].

The topology can also be given by means of an increasing sequence $(||\cdot||_k)_{k\in\mathbb{N}}$ of semi-norms. Then the function

$$d(x,y) = \sum_{k=1}^{\infty} 2^{-k} \frac{||x-y||_k}{1+||x-y||_k}, \quad x, y \in \mathbb{R}^{\mathbb{N}}$$

defines an equivalent metric. Let $\Pi_k : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^k$ be the projection on the first k coordinates, $\Pi_k(x) = (x_1, \ldots, x_k)^T$. Then we can take $||x||_k = ||\Pi_k(x)||^{\mathbb{R}^k}$ as the standard norm in $\mathbb{R}^k$, $k \in \mathbb{N}$. Since convergence in $\mathbb{R}^{\mathbb{N}}$ is coordinatewise, we have the following:

Proposition 2.1. *The sequence $\{x^{(n)}\}_{n\in\mathbb{N}}$ of elements of $\mathbb{R}^{\mathbb{N}}$ tends to 0 if and only if $\forall\ k \in \mathbb{N} : \lim\limits_{n\to\infty} ||x^{(n)}||_k = 0$.*

In the next parts of this article we need the following sufficient condition of convergence of a sequence in $\mathbb{R}^{\mathbb{N}}$ to the origin.

Proposition 2.2. *Let $\{x^{(n)}\}_{n\in\mathbb{N}}$ be a sequence of elements from $\mathbb{R}^{\mathbb{N}}$ such that $\forall\ k \in \mathbb{N}\ \exists\ n(k) \in \mathbb{N} : \forall\ n > n(k)\ x_i^{(n)} = 0$ for $i = 1, \ldots, k$. Then $\{x^{(n)}\}_{n\in\mathbb{N}}$ is convergent to the origin in $\mathbb{R}^{\mathbb{N}}$.*

Proof. Let $k \in \mathbb{N}$. From the assumption, there is $n(k)$ such that for all $n > n(k)$, $\Pi_k\left(x^{(n)}\right) = 0 \in \mathbb{R}^k$. Hence: $\lim\limits_{n\to\infty} ||x^{(n)}||_k = 0$, so by Proposition 2.1: $\{x^{(n)}\}_{n\in\mathbb{N}}$ tends to zero in $\mathbb{R}^{\mathbb{N}}$. □

The fact that in the infinite-dimensional space $\mathbb{R}^{\mathbb{N}}$ we work with product topology has the following consequence:

Proposition 2.3. [11]
Let $K \subset \mathbb{R}^{\mathbb{N}}$. The set K is relatively compact in $\mathbb{R}^{\mathbb{N}}$ if and only if for each $n \in \mathbb{N}$ there is $l_n > 0$ such that

$$|x_n| \le l_n, \ \textit{for each } x = (x_1, x_2, \ldots) \in K.$$

In other words, the set K to be relatively compact must be contained in some parallelepiped in $\mathbb{R}^{\mathbb{N}}$.

Corollary 2.4. *A compact subset K of $\mathbb{R}^{\mathbb{N}}$ is nowhere dense, i.e. $intK = \emptyset$.*

Proof. Let $x = (x_1, x_2, \ldots) \in K$ and $|x_n| \leq l_n$ for $n \in \mathbb{N}$. Suppose $intK \neq \emptyset$. Then there is a ball B, in the metric ρ, contained in K. We can assume that B has the center at 0 and the radius ε. Then for $n > \log_2 \frac{1+l_n}{\varepsilon(2+l_n)}$ the point $x = (0, \ldots, l_n + 1, 0, \ldots)$, with $x_n = l_n + 1$, belongs to B, but does not belong to K. This gives contradiction. □

Let $L_c(\mathbb{R}^{\mathbb{N}}, \mathbb{R})$ be the space of all linear and continuous mappings from $\mathbb{R}^{\mathbb{N}}$ to $\mathbb{R}$.

Theorem 2.5. ([1])
$L_c(\mathbb{R}^{\mathbb{N}}, \mathbb{R}) \approx \mathbb{R}^{(\mathbb{N})}$.

As in $\mathbb{R}^{\mathbb{N}}$ not every linear subspace is closed, we need the following useful fact:

Proposition 2.6. *Let Y be a linear subspace of $\mathbb{R}^{\mathbb{N}}$. Then for all $v \in L_c(\mathbb{R}^{\mathbb{N}}, \mathbb{R})$ the following holds: $v(Y) = 0 \Leftrightarrow v(\overline{Y}) = 0$, (where $\overline{Y}$ denotes the closure of Y).*

Each element $A \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ may be interpreted as a function $A : \mathbb{N} \times \mathbb{N} \ni (i, j) \mapsto a_{ij} \in \mathbb{R}$, and it is called *an infinite matrix.* We write then $A = (a_{ij})_{i,j \in \mathbb{N}}$. By $E = (\delta_{ij})_{i,j \in \mathbb{N}}$, where $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ii} = 1$, we denote the identity matrix. We shall deal with differential systems described by infinite matrices of some particular type.

Definition 2.7. We say that $A = (a_{ij})_{i,j \in \mathbb{N}}$ is
a) *row-finite* if for each $i \in \mathbb{N}$ there is $\alpha(i) \in \mathbb{N} : a_{ij} = 0$ for $j > \alpha(i)$,
b) *column-finite* if A^T is row-finite,
c) *lower-diagonal* if $a_{ij} = 0$ for $j > i$,
d) *upper-diagonal* if $a_{ij} = 0$ for $j < i$.

Of course, a lower-diagonal matrix is a particular case of row-finite matrix. Each of these sets of matrices forms an algebra over $\mathbb{R}$ with a unit $E = (\delta_{ij})_{i,j \in \mathbb{N}}$ (in particular multiplication is associative). Hence the powers $A^k, k \in \mathbb{N} \cup \{0\}$, of the matrix A of one of these types are of the same type.

Let $A = (a_{ij})_{i \in \mathbb{N}, j \in \mathbb{N}}$ be an infinite matrix. Then by $A_{\underline{n}}$ we denote the matrix derived from A by replacing by 0 all elements except those occupying the first n rows. By $A_{n|}$ we denote the matrix derived from A by replacing by 0 all elements except those occupying the first n columns. Then $(A_{\underline{n}})_{m|} = (A_{m|})_{\underline{n}}$ have the same elements a_{ij} as A for $i \leq m$ and $j \leq n$. If $m = n$ we write $(A_{\underline{n}})_{n|} = A_{(n)}$. The matrices $A_{\underline{n}}$, $A_{n|}$ and $A_{(n)}$ are infinite, but they have properties similar to properties of finite matrices. For $A = (a_{ij}) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ let us denote $A_{[n]} = (a_{ij})_{i \leq n, j \leq n}$. In this way we cut off from the matrix A first n rows and columns.

For upper-diagonal and lower-diagonal matrices we have the following direct properties, very useful for our applications.

Proposition 2.8. *Let $A, B \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ be both lower-diagonal or upper-diagonal matrices. Then:*
a) $\forall\ k \in \mathbb{N} : \left(A_{(n)}\right)^k = (A^k)_{(n)}$ and $\left(A_{[n]}\right)^k = (A^k)_{[n]}$.
b) $(AB)_{(n)} = A_{(n)}B_{(n)}$ and $(AB)_{[n]} = A_{[n]}B_{[n]}$.
c) Let $w_n(\lambda) = \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + a_0$ be the characteristic polynomial of the matrix $A_{[n]}$. Then $A^n_{(n)} + c_{n-1}A^{n-1}_{(n)} + \cdots + c_1 A_{(n)} + c_0 E_{(n)} = 0$.

If in the previous proposition we take only one infinite column $b \in \mathbb{R}^{\mathbb{N}}$ instead of the infinite matrix B, we get the following corollary.

Corollary 2.9.
If $b \in \mathbb{R}^{\mathbb{N}}$ is a column and A is a lower-diagonal matrix, then $(Ab)_{\underline{n}} = A_{\underline{n}} b_{\underline{n}} = A_{(n)} b_{\underline{n}}$ and $\Pi_n(Ab) = A_{[n]}\Pi_n(b)$.

Proposition 2.10. *Let $b \in \mathbb{R}^{(\mathbb{N})}$ and $b = (b_1, b_2, \ldots, b_n, 0, \ldots)^T$, and let A be an infinite matrix. Then $Ab = A_{n|}b$ and $b^T A = b^T A_{\underline{n}}$.*

Definition 2.11. ([6])
Let $A = (a_{ij})_{i\in\mathbb{N}, j\in\mathbb{N}}$ be an infinite matrix and suppose that there is $r > 0$ such that for each $i, j \in \mathbb{N}$ the power series $\sum\limits_{k=0}^{\infty} \frac{t^k}{k!} \left(A^k\right)_{ij}$ has the radius of convergence greater or equal $r > 0$. Then we define the matrix e^{tA} by $\left(e^{tA}\right)_{ij} = \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} \left(A^k\right)_{ij}$.

From the above definition and from Proposition 2.8 and Corollary 2.9 we have the following proposition for lower-diagonal matrices.

Proposition 2.12. *Let $A = (a_{ij})_{i\in\mathbb{N}, j\in\mathbb{N}}$ be a lower-diagonal matrix. Then:*
a) the product $A^k x^0$ exists for all $k \in \mathbb{N} \cup \{0\}$ and $x^0 \in \mathbb{R}^{\mathbb{N}}$.
b) e^{tA} exists for each $t \geq 0$ and it is lower-diagonal.
c) the function $x(t) = e^{tA}x^0$ is the unique analytic solution of the lower-diagonal problem:

$$\dot{x}(t) = Ax(t),\ x(0) = x^0$$

for every $x^0 \in \mathbb{R}^{\mathbb{N}}$.

Proof. The parts a) and b) are the direct consequences of that A is a lower-diagonal matrix.

We prove the part c). For each $i \in \mathbb{N}$ we show that $\frac{d}{dt}x_i(t) = \left(\frac{d}{dt}e^{tA}x^0\right)_i = \left(Ae^{tA}x^0\right)_i$, where the subscript i denotes the i-th coordinate. Let A_i denotes the i-th row of the matrix A. Then $(A^{k+1})_i = A_i A^k$. Hence $\left(\frac{d}{dt}e^{tA}x^0\right)_i = \frac{d}{dt}\sum\limits_{k=0}^{\infty}\frac{t^k}{k!}(A^k)_i x^0 = \sum\limits_{k=0}^{\infty}\frac{t^k}{k!}(A^{k+1})_i x^0 = A_i e^{tA}x^0 = \left(Ae^{tA}x^0\right)_i$. ∎

Corollary 2.13. *Let A be a lower-diagonal matrix and $b \in \mathbb{R}^{\mathbb{N}}$. Then for each $n \in \mathbb{N}$: $\left(e^{tA}\right)_{[n]} = e^{tA_{[n]}}$ and $\Pi_n\left(e^{tA}b\right) = e^{tA_{[n]}}\Pi_n(b)$.*

Proposition 2.14. *Let A be a lower-diagonal matrix, $x^0 \in \mathbb{R}^{\mathbb{N}}$ and $G^{(n)}(t) = e^{tA_{(n)}}x^0$. Then for all $t \geq 0$ and $n > 0$ we have the following:*
a) the series $G^{(n)}(t)$ is convergent and $\lim_{n\to\infty} G^{(n)}(t) = e^{tA}x^0$.
b) the function $t \mapsto \Pi_n\left(G^{(n)}(t)\right)$ is the unique analytic solution of the n-dimensional problem:

$$\dot{z}(t) = A_{[n]}z(t),\ z(0) = \Pi_n(x^0).$$

Proof. It follows from properties of finite-dimensional matrices and from Propositions 2.8 and 2.2. □

We formulate similar facts for upper-diagonal matrices. The next propositions follows from Proposition 2.8 and 2.10.

Proposition 2.15. *Let $A = (a_{ij})_{i\in\mathbb{N},j\in\mathbb{N}}$ be an upper-diagonal matrix. Then:*
a) the product A^kx^0 exists for all $k \in \mathbb{N}\cup\{0\}$ and $x^0 \in \mathbb{R}^{(\mathbb{N})}$.
b) e^{tA} exists for each $t \geq 0$ and it is upper-diagonal.
c) the function $x(t) = e^{tA}x^0$ is the solution of the upper-diagonal problem:

$$\dot{x}(t) = Ax(t),\ x(0) = x^0$$

for every $x^0 \in \mathbb{R}^{(\mathbb{N})}$.

Corollary 2.16. *Let A be upper-diagonal and $x^0 = (x_1^0, \ldots, x_n^0, 0, \ldots)^T \in \mathbb{R}^{(\mathbb{N})}$. Then $e^{tA}x^0 = e^{tA_{(n)}}x^0$.*

Proposition 2.17. *Let A be an upper-diagonal matrix, $x^0 = (x_1^0, \ldots, x_n^0, 0, \ldots)^T \in \mathbb{R}^{(\mathbb{N})}$ and $G^{(n)}(t) = e^{tA_{(n)}}x^0$. Then for all $t \geq 0$ and $n > 0$ we have the following:*
a) the series $G^{(n)}(t)$ is convergent and $\lim_{n\to\infty} G^{(n)}(t) = e^{tA}x^0$.
b) the function $t \mapsto \Pi_n\left(G^{(n)}(t)\right)$ is the unique analytic solution of the n-dimensional problem:

$$\dot{z}(t) = A_{[n]}z(t),\ z(0) = (x_1^0, x_2^0, \ldots, x_n^0)^T.$$

Example 2.18. Let A be the upper-diagonal matrix of the following form:
$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$. Then the exponential matrix exists and:

$$e^{tA} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} & \ldots \\ 0 & 1 & t & \frac{t^2}{2!} & \ldots \\ 0 & 0 & 1 & t & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.1}$$

Now let us consider the upper-diagonal initial value problem $\dot{x}_i = x_{i+1}, i \in \mathbb{N}$, and $x^0 = (x_1^0, x_2^0, \ldots, x_n^0, 0, \ldots) \in \mathbb{R}^{(\mathbb{N})}$. Then the corresponding solution is analytic, unique and has the following form:

$$x(t) = \begin{pmatrix} x_1^0 + x_2^0 t + \cdots + x_n^0 \frac{t^{n-1}}{s!} \\ x_2^0 + \cdots + x_n^0 \frac{t^{n-1}}{n!} \\ \vdots \\ x_n^0 \\ 0 \\ \vdots \end{pmatrix} = e^{tA}x^0 \in \mathbb{R}^{(\mathbb{N})}, \text{ where } e^{tA} \text{ has the form (2.1)}$$

and for $x^0 \in \mathbb{R}^{(\mathbb{N})}$: $e^{tA}x^0 = e^{tA_{(n)}}x^0$.

3 Lower-diagonal systems and controllability

By a lower-diagonal system we mean the dynamical system $\dot{x}(t) = Ax(t)$, where A is an infinite lower-diagonal matrix. Lower-diagonal systems are the only ones, for which most facts known for finite systems hold true. In particular, for every $x^0 \in \mathbb{R}^{\mathbb{N}}$ the initial value problem $\dot{x}(t) = Ax(t), x(0) = x^0$ has the unique solution $x(t) = e^{tA}x_0$, by Proposition 2.12.

Let us consider the control system

$$(\Lambda) : \dot{x}(t) = Ax(t) + bu(t),\ x(0) = x^0 \in \mathbb{R}^{\mathbb{N}}, \tag{3.1}$$

where A is a lower-diagonal infinite matrix from $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and $b \in \mathbb{R}^{\mathbb{N}}$. The controls $u(\cdot)$ are assumed to be real-valued, locally Lebesgue-integrable functions.

Proposition 3.1. *For any $x^0 \in \mathbb{R}^{\mathbb{N}}$ the trajectory of the system (Λ) corresponding to initial condition $x(0) = x^0$ and control u exists and has the form:* $\gamma(t, x^0, u) = e^{tA}x^0 + \int\limits_0^t e^{(t-\tau)A}bu(\tau)d\tau.$

Proof. We show that for each $i \in \mathbb{N}$: $\left(\frac{d}{dt}\gamma(t, x^0, u)\right)_i = A_i\gamma(t, x^0, u) + b_iu(t)$, where the subscript i denotes the i-th coordinate. Similarly as in the proof of Proposition 2.12 we have that $\left(\frac{d}{dt}e^{tA}b\right)_i = A_ie^{tA}b = \left(Ae^{tA}b\right)_i$. Additionally $\left(\frac{d}{dt}\int\limits_0^t e^{(t-\tau)A}bu(\tau)d\tau\right)_i = \frac{d}{dt}\int\limits_0^t \left(e^{(t-\tau)A}b\right)_i u(\tau)d\tau = b_iu(t) + A_i\int\limits_0^t e^{(t-\tau)A}bu(\tau)d\tau.$
Hence $\forall\ (i \in \mathbb{N}) : \left(\frac{d}{dt}\gamma(t, x^0, u)\right)_i = A_i\left(e^{tA}x^0 + \int\limits_0^t e^{(t-\tau)A}bu(\tau)d\tau\right) + b_iu(t).$ □

Definition 3.2. Let $\{\Lambda_n\}_{n\in\mathbb{N}}$ be the sequence of the following systems

$$(\Lambda_n) : \dot{x}(t) = A_{(n)}x(t) + b_{\underline{n}}u(t),\ x(0) = x^0 \in \mathbb{R}^{\mathbb{N}}, \tag{3.2}$$

where $A_{(n)}$ is the infinite matrix derived from the matrix A of the system (Λ) and $b_{\underline{n}}$ is got from the column b of (Λ). By $F_n(\cdot, x^0, u)$ we denote the solution of the system (Λ_n) corresponding to the initial condition $x(0) = x^0$ and the control u.

Let us consider the sequence $\{F_n(\cdot, x^0, u)\}_{n\in\mathbb{N}}$, where $F_n(t, x^0, u) = e^{tA_{(n)}}x^0 + \int_0^t e^{(t-\tau)A_{(n)}} b_{\underline{n}} u(\tau)d\tau$. By definitions of $A_{(n)}$ and $b_{\underline{n}}$ we have that for all n : $F_n(t, x^0, u) \in \mathbb{R}^{(\mathbb{N})}$. From Proposition 2.14 we get the following:

Proposition 3.3. *Let $\gamma(\cdot, x^0, u)$ be the solution of the initial value problem (3.1). Then for all $t \geq 0$: $\gamma(t, x^0, u) = \lim_{n\to\infty} F_n(t, x^0, u)$.*

Definition 3.4. By $\Re_t(0)$ we denote the *attainable set in time $t \geq 0$ from the initial state $x(0) = 0$*, i.e.

$$\Re_t(0) = \{x \in \mathbb{R}^{\mathbb{N}} : x = \int_0^t e^{(t-\tau)A} bu(\tau)d\tau,\ u \in L_1([0, t], \mathbb{R})\}. \tag{3.3}$$

Remark 3.5. Let $0 \leq t_1 \leq t_2$. Then $\Re_{t_1}(0) \subseteq \Re_{t_2}(0)$.

Definition 3.6. Let Y be a linear subspace of $\mathbb{R}^{\mathbb{N}}$. The system (Λ) is said to be *approximately controllable from the origin in time t on Y* if $Y \subseteq \overline{\Re_t(0)}$. The system (Λ) is said to be *exactly controllable from the origin in time t on Y* if $Y \subseteq \Re_t(0)$.

The system (Λ) is said to be *globally approximately controllable from the origin in time t* if $\overline{\Re_t(0)} = \mathbb{R}^{\mathbb{N}}$. The system (Λ) is said to be *globally exactly controllable from the origin in time t* if $\Re_t(0) = \mathbb{R}^{\mathbb{N}}$.

The system (Λ) is said to be *globally approximately controllable from the origin in finite time* if there is $t > 0$ such that (Λ) is globally approximately controllable from the origin in time t.

Let $P_t : L_1([0, t], \mathbb{R}) \to \mathbb{R}^{\mathbb{N}}$ be the operator associated with (3.1) given by the formula:

$$P_t u = \int_0^t e^{(t-\tau)A} bu(\tau)d\tau \tag{3.4}$$

and defined for arbitrary, but fixed, $t > 0$. Then $\Re_t(0) = \operatorname{im} P_t$.

Proposition 3.7. *For all $t > 0$ the mapping P_t is linear and continuous.*

Proof. The linearity is obvious. To prove continuity let us first observe that for each $k \in \mathbb{N}$: $||P_t u||_k = ||\int_0^t e^{(t-\tau)A_{[k]}} \Pi_k(b)u(\tau)d\tau||^{\mathbb{R}^k}$, from Corollary 2.13.

As the function $\tau \mapsto e^{(t-\tau)A_{[k]}} \Pi_k(b)$ is of class L_∞ on $[0, t]$, the operators $u \mapsto \int_0^t e^{(t-\tau)A_{[k]}} \Pi_k(b)u(\tau)d\tau \in \mathbb{R}^k$ are continuous, by [11]. Hence for $(u_n)_{n\in\mathbb{N}}$ such that $\lim_{n\to\infty} u_n = 0$ we have that $\lim_{n\to\infty} ||P_t u_n||_k = 0$. From Proposition 2.2 we get that $\lim_{n\to\infty} P_t u_n = 0$. So P_t is continuous. □

Proposition 3.8. *The operator P_t is compact.*

Proof. As the proof is similar as in [24] so we give only its sketch.

Firstly it is proved that P_t is compact for L_∞-controls. For this we need the relative compactness in $\mathbb{R}^\mathbb{N}$ of the set: $M = \bigcup\limits_{0\le\tau\le t}\ \bigcup\limits_{|u(\tau)|\le 1} \{e^{(t-\tau)A}bu(\tau)\}$, where $u(\cdot)$ is any measurable control function. This can be shown as in Lemma 2.1 in [24]. Here we need the continuity of the operator P_t shown in Proposition 3.7.

Then it is proved that P_t is compact for L_∞ controls. In our case when the image of P_t is a subset of a Frechet space we can use instead of Mazur's theorem (for Banach spaces) the fact that the convex hull of a relatively compact set in a locally convex space is relatively compact, then the closed convex hull of it is compact.

In the next step using the integration by parts formula and the properties of the lower-diagonal matrix A and its exponential matrix it is proved that P_t is compact for L_1-controls. □

Proposition 3.9. *The dynamical system* (Λ) *is not globally exactly controllable from the origin in any finite time* $t \ge 0$.

Proof. Let $B_n := \{u : \int\limits_0^t |u(\tau)|dt < n\}$ and $P_t(B_n)$ be the image of B_n under the operator P_t. Then by Proposition 3.8 the set $P_t(B_n)$ is a relatively compact set in $\mathbb{R}^\mathbb{N}$, so it is nowhere dense in $\mathbb{R}^\mathbb{N}$ by Corollary 2.4. Hence by the Baire category theorem, [12], we have that $\mathbb{R}^\mathbb{N} \ne \bigcup\limits_{n=1}^{\infty} P_t(B_n)$.

As $\Re_t(0) \subseteq \bigcup\limits_{n=1}^{\infty} P_t(B_n)$, so ($\Lambda$) is not globally exactly controllable from the origin in any finite time $t \ge 0$ (t in the proof is arbitrary). □

Example 3.10. Let

$$(\Lambda) : \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 \\ \vdots \\ \dot{x}_n = x_{n-1} \\ \vdots \end{cases}$$

Then for each $n \in \mathbb{N}$ the corresponding (Λ_n) is exactly controllable from the origin in any time t on $Y_n = \{x \in \mathbb{R}^\mathbb{N} : \ \forall\ k > n,\ x_k = 0\}$. This follows from the controllability notion for finite-dimensional systems. Now we show that (Λ) is globally approximately controllable. Let $\tilde{x} \in \mathbb{R}^\mathbb{N}$. Firstly let us notice that for each $k \in \mathbb{N}$ there is the control function $u_k(\cdot)$ such that for all $n \in \mathbb{N}$: $\Pi_k(F_n(t, 0, u_k)) = \tilde{x}_{\underline{k}}$.

Now let us consider the sequence $\left(x^{(k)}\right)_{k\in\mathbb{N}}$, where $x^{(k)} = \gamma(t, 0, u_k) = \lim\limits_{n\to\infty} F_n(t, 0, u_k)$. Then $\lim\limits_{k\to\infty} x^{(k)} = \tilde{x}$. The convergence follows from Proposition 2.2.

Proposition 3.11. *For all* $t > 0 : \overline{\Re_t(0)} = \overline{\operatorname{span}\{b, Ab, \dots\}}$.

Proof. To show equality of closed spaces in $\mathbb{R}^{\mathbb{N}}$ it is enough to show that they are annihilated by the same vectors from $\mathbb{R}^{(\mathbb{N})}$.
Let $t_0 > 0$ and $v \in \mathbb{R}^{(\mathbb{N})}$. Assume that $v^T\overline{\Re_{t_0}(0)} = 0$. Then, by Remark 3.5, for every $0 \leq t \leq t_0 : v^T\overline{\Re_t(0)} = 0$ and so $v^T\gamma(t,0,u) = 0$ for each u. Let us take $u \equiv 1$. Then for every $0 \leq t \leq t_0$: $\int_0^t v^T e^{(t-\tau)A} b d\tau = 0$. Hence $\frac{d^k}{dt^k}_{|t=0} \int_0^t v^T e^{(t-\tau)A} b d\tau = 0$. Therefore for all $k \geq 0 :\ v^T A^k b = 0$. So we have $v^T\overline{\text{span}\{b, Ab, \ldots\}} = 0$.

Now let $v \in \mathbb{R}^{(\mathbb{N})}$ and $v^T\overline{\text{span}\{b, Ab, \ldots\}} = 0$. Then $v^T \left(\lim_{n\to\infty} \sum_{k=0}^n A^k b \frac{t^k}{k!}\right) = 0$ for all $t \geq 0$. Hence $v^T \sum_{k=0}^\infty A^k b \frac{t^k}{k!} = 0$, i.e. $v^T e^{tA} b = 0$. Since $\gamma(t,0,u) = \int_0^t e^{(t-\tau)A} b u(\tau) d\tau$, then $\forall\ t \geq 0,\ v^T\gamma(t,0,u) = 0$. This implies $v^T\overline{\Re_t(0)} = 0$ for all $t \geq 0$. □

Proposition 3.12. *The system* (Λ) *is globally approximately controllable in finite time if and only if (for* $v \in \mathbb{R}^{(\mathbb{N})} : v^T(b, Ab, A^2b, \ldots) = 0 \Rightarrow v = 0$*).*

Proof. Λ is globally approximately controllable in finite time iff there is $t > 0$ such that (Λ) is globally approximately controllable in time t. From Proposition 3.11 this is equivalent to $\overline{\text{span}\{b, Ab, \ldots\}} = \mathbb{R}^{\mathbb{N}}$. This means that for $v \in \mathbb{R}^{(\mathbb{N})} : v^T\overline{\text{span}\{b, Ab, \ldots\}} = 0 \iff v = 0$. Hence, from Remark **??**, this is equivalent to $v^T\{b, Ab, \ldots\} = 0 \iff v = 0$. □

Proposition 3.13. (Λ) *is globally approximately controllable in finite time if and only if*

$$\forall n \in \mathbb{N}\ \mathit{rank}\begin{pmatrix} b, Ab, \ldots, A^{n-1}b \end{pmatrix}_{\underline{n}} = n.$$

Proof. Let (Λ) be not globally approximately controllable. Then from Proposition 3.12 there is $v \neq 0$, $v \in \mathbb{R}^{(\mathbb{N})}$, such that $v^T(b, Ab, A^2b, \ldots) = 0$. Let $v^T = (v_1, \ldots, v_n, 0 \ldots)$. Then $v^T(b, Ab, A^2b, \ldots) = v^T(b, Ab, A^2b, \ldots)_{\underline{n}} = 0$. Since A is lower-diagonal, from Remark 2.9, $v^T(b_{\underline{n}}, A_{(n)}b_{\underline{n}}, A^2_{(n)}b_{\underline{n}}, \ldots) = 0$. Then $\text{rank}(b_{\underline{n}}, A_{(n)}b_{\underline{n}}, A^2_{(n)}b_{\underline{n}}, \ldots) < n$. Hence also
$\text{rank}(b_{\underline{n}}, A_{(n)}b_{\underline{n}}, A^2_{(n)}b_{\underline{n}}, \ldots, A^{n-1}_{(n)}b_{\underline{n}}) < n$ and $\text{rank}\begin{pmatrix} b, Ab, \ldots, A^{n-1}b \end{pmatrix}_{(n)} < n$.

Let now $\text{rank}\begin{pmatrix} b, Ab, \ldots, A^{n-1}b \end{pmatrix}_{(n)} < n$. Using Proposition 2.8 c) we can reverse all the above steps and get that (Λ) is not globally approximately controllable. □

Corollary 3.14. (Λ) *is globally approximately controllable if and only if for all* $n \in \mathbb{N}$ *the systems* (Λ_n) *are exactly controllable on* $Y_n = \{(x_1, x_2, \ldots, x_n, 0, \ldots) : x_i \in \mathbb{R}\}$.

Example 3.15. The system from Example 3.10 is globally approximately controllable and the matrix $\begin{pmatrix} b, Ab, \ldots, \end{pmatrix} = E$. Additionally, for any $n \in \mathbb{N} : (\Lambda_n)$ is exactly controllable on Y_n.

Remark 3.16. If for some system (Λ) $\text{rank}(b, Ab, \ldots) = \infty$, this does not imply that (Λ) is globally approximately controllable.

In the next proposition we formulate a sufficient condition of approximate controllability for systems with lower-diagonal and column-finite matrix A. Recall that if A and b are column-finite, then also for every $k \in \mathbb{N}$, $A^k b$ is column-finite.

Proposition 3.17. *Let* (Λ) *be the system with lower-diagonal column finite matrices and let:*

$$\forall n \in \mathbb{N}\ \exists k \in \mathbb{N} \cup \{0\} : \ \operatorname{rank}\left(b, Ab, \ldots A^k b\right) = \operatorname{rank}\left(b, Ab, \ldots A^k b, e_n\right), \quad (3.5)$$

where e_n *denotes the infinite column with* 1 *at the* n*-th position. Then* (Λ) *is globally approximately controllable from the origin in finite time.*

Proof. The condition (3.5) means that $\mathbb{R}^{(\mathbb{N})} \subseteq \operatorname{span}\{b, Ab, \ldots\}$. Then from Proposition 3.11, (Λ) is globally approximately controllable. □

Now we give an example that shows why the condition in Proposition 3.17 is only sufficient.

Example 3.18. Let $(\Lambda) : \begin{cases} \dot{x}_1 = u \\ \dot{x}_2 = x_1 + u \\ \dot{x}_k = x_{k-1}, \quad k \geq 3 \end{cases}$

Then $(b, Ab, \ldots) = \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 1 & 1 & 0 & \ddots \\ 0 & 1 & 1 & \ddots \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \end{pmatrix}$ and (Λ) is globally approximately controllable, from Proposition 3.13. But the condition (3.5) is not satisfied.

4 Row-finite systems and observability

If A is row–finite and is not lower diagonal, we can lose uniqueness of smooth solutions and e^{At} may not exist.

Let A be row-finite. Then $A^k x^0$ exists for all $x^0 \in \mathbb{R}^{\mathbb{N}}$. Let $(A^k)_i$ be the i-th row of the row-finite matrix A^k. Then the value of $(A^k)_i x^0$ is a finite sum for each $x^0 \in \mathbb{R}^{\mathbb{N}}$. However the series $\Gamma_{x^0,A} := \sum\limits_{k=0}^{\infty} \frac{t^k}{k!} A^k x^0$ may not be convergent as in Example 4.3 below.

Proposition 4.1. *Let* A *be a row–finite matrix. Then for all* $x^0 \in \mathbb{R}^{\mathbb{N}}$ *the initial value problem* $\dot{x}(t) = Ax(t),\ x^0 \in \mathbb{R}^{\mathbb{N}}$ *has the unique formal solution given by the formal power series* $\Gamma_{x^0,A} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k x^0$.

Proof. Observe that $\frac{d}{dt}\Gamma_{x^0,A} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^{k+1} x^0 = A\Gamma_{x^0,A}$. □

If $\Gamma_{x^0,A}$ is convergent then we have an analytic solution. The discussion of existence and uniqueness of solutions of the initial value problem (1.1) in the case when A is row-finite can be found, e.g., in [6]. We recall Theorem 6.2 of [6].

Proposition 4.2. *Let* $A = (a_{ij})_{i,j\in\mathbb{N}}$. *Assume that there exists a sequence* $(S_n)_{n\in\mathbb{N}}$ *of finite subsets of* $\mathbb{N}$ *such that for each* $n \in \mathbb{N} : S_n \subset S_{n+1}$, $\sum_{i\in\mathbb{N}} S_n = \mathbb{N}$ *and* $\{j \in \mathbb{N} : a_{ij} \neq 0\} \subset S_n$ *for* $i \in S_n$. *Then the initial value problem:* $\dot{x}(t) = Ax(t)+b,\ x(0) = x^0$ *has a unique solution. Otherwise, the problem (1.1) has infinitely many solutions.*

Example 4.3. Let $\frac{dx_i}{dt}(t) = x_{i+1}(t),\ i \in \mathbb{N}$ and $x(0) = x^0 = (c_1, c_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$. The formal solution can be written in the following way: $\left\{\sum_{k=0}^{\infty} c_{i+k}\frac{t^k}{k!},\right\}_{i\in\mathbb{N}}$. On the other hand a smooth solution is produced by an arbitrary smooth function $\varphi = \varphi(t)$ such that $\frac{d^k\varphi}{dt^k}(0) = c_{k+1}, k = 0, 1, 2, \ldots$, and $x_k(t) = \frac{d^k\varphi}{dt^k}(t),\ k = 0, 1, 2, \ldots$. Since there are infinitely many such functions (they differ by "flat" functions with all derivatives at $t = 0$ equal 0), we have infinitely many smooth solutions. In spite of that the product $e^{tA}x^0$ may not exist as e^{tA} is not row-finite. Observe that the condition of Proposition 4.2 does not hold.

Now we formulate observability conditions for systems with outputs that are described by row-finite matrices. The most part of this material was proved in [3].

We are concerned with the system with output:

$$(\Sigma): \begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ y(t) & = & Cx(t), \end{array} \tag{4.1}$$

where $x : [0, \infty) \to \mathbb{R}^{\mathbb{N}}, y : [0, \infty) \to \mathbb{R}^r$, and $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ and $C \in \mathbb{R}^{r\times\mathbb{N}}$ are row–finite. Let $x^0 \in \mathbb{R}^{\mathbb{N}}$. Given a formal solution $\Gamma_{x^0,A}$ of the dynamical part of the system and corresponding to the initial condition x^0 we define the formal output: $\mathcal{Y}_{x^0} = C\Gamma_{x^0,A}$.

Definition 4.4. We say that $x^1, x^2 \in \mathbb{R}^{\mathbb{N}}$ are *indistinguishable* (with respect to (Σ)) if $\mathcal{Y}_{x^1} = \mathcal{Y}_{x^2}$. Otherwise x^1, x^2 are distinguishable. We say that the system (Σ) is *observable* if any two distinct points are distinguishable.

Let Y be a linear subspace of $\mathbb{R}^{\mathbb{N}}$. We say that (Σ) *is observable on* Y if two different points from Y are distinguishable.

Proposition 4.5. ([3])
The points $x^1, x^2 \in \mathbb{R}^{\mathbb{N}}$ *are indistinguishable iff for all* $k \in \mathbb{N} \cup \{0\} : CA^kx^1 = CA^kx^2$.

Let $D = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix}$ and $\mathcal{D}(x) = Dx, \mathcal{D} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$. From Proposition 4.5 we get a similar characterization of observability as in the finite-dimensional case.

Proposition 4.6. ([3])
a) (Σ) *is observable* $\Longleftrightarrow \mathcal{D}$ *is injective.*
b) (Σ) *is observable on* $Y \Longleftrightarrow \mathcal{D}_{|Y}$ *is injective.*

Corollary 4.7. ([3])
System (Σ) *is observable if and only if* $\forall\, n \in \mathbb{N}\ \exists k \in \mathbb{N} \cup \{0\}$:

$$\operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \end{pmatrix} = \operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \\ e_n^T \end{pmatrix},$$

where e_n^T *denotes the infinite row with* 1 *at the* n*-th position.*

Since the rows of D correspond to derivatives of the output, one can characterize observability as possibility to compute every state variable as a linear combination of finitely many outputs and their derivatives.

Example 4.8.
a) The system $\left\{ \begin{array}{l} \dot{x}_i = x_{i+1},\ i \in \mathbb{N} \\ y = x_1, \end{array} \right.$, $x \in \mathbb{R}^{\mathbb{N}}$,
is observable.
b) The system $\left\{ \begin{array}{l} \dot{x}_i = x_{i+1},\ i \in \mathbb{N} \\ y = x_1 + x_2, \end{array} \right.$, $x \in \mathbb{R}^{\mathbb{N}}$,
is not observable, because the mapping $\mathcal{D} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$, where $\mathcal{D}x = Dx$ and
$D = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots\ddots & \ddots & \ddots & \ddots & \end{pmatrix}$, does not distinguish points $(0, 0, ...)$ and $(1, -1, 1, -1, \ldots)$. But if we consider $\mathcal{D}_{|\mathbb{R}^{(\mathbb{N})}} = \tilde{\mathcal{D}} : \mathbb{R}^{(\mathbb{N})} \to \mathbb{R}^{(\mathbb{N})}$, then we get an injective mapping. Hence every of two finite sequences are distinguishable and the system is observable on $\mathbb{R}^{(\mathbb{N})}$.

Proposition 4.9. *The system* (Σ) *is observable on* $\mathbb{R}^{(\mathbb{N})}$ *if and only if*

$$\forall n \in \mathbb{N}\ \exists k \in \mathbb{N} \cup \{0\} : \quad \operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \end{pmatrix}_{n|} = n. \tag{4.2}$$

Proof. Let $n \in \mathbb{N}$ be such that for all $k \in \mathbb{N}$: $\operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \end{pmatrix}_{n|} < n$. This is equivalent to $\operatorname{rank} D_{n|} = \operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \\ \vdots \end{pmatrix}_{n|} < n$. This means that there is

$v \neq 0$ and $v = (v_1, v_2, \dots, v_n, 0, \dots)^T$ such that $D_{n|}v = 0$, which is equivalent to $Dv = 0$. This holds if and only if (Σ) is not observable on $\mathbb{R}^{(\mathbb{N})}$. □

Let us observe that the system from Example 4.8 b) satisfies the condition (4.2).

5 Upper-diagonal systems and dualities

Let us consider the system with output:

$$(\Sigma\text{up}) : \begin{array}{rcl} \dot{x}(t) & = & Ax(t) \\ y(t) & = & cx(t), \end{array} \tag{5.1}$$

on the space $X = \mathbb{R}^{(\mathbb{N})}$ and $y : [0, \infty) \to \mathbb{R}$. Assume that the matrix $A \in \mathbb{R}^{\mathbb{N}\times\mathbb{N}}$ is upper-diagonal and $c^T \in \mathbb{R}^{(\mathbb{N})}$ so c is row-finite. Let $x^0 \in \mathbb{R}^{(\mathbb{N})}$. Then the corresponding output is well defined and $y(t) = ce^{tA}x^0,\ t \geq 0$. The definition of observability is similar to those given by the second part of Definition 4.4 but now we use analytic solutions corresponding to initial conditions from the space $\mathbb{R}^{(\mathbb{N})}$.

Definition 5.1. We say that $x^1, x^2 \in \mathbb{R}^{(\mathbb{N})}$ are *indistinguishable* (with respect to (Σ_{up})) if for all $t \geq 0 : ce^{tA}x^1 = ce^{tA}x^2$. Otherwise $x^1, x^2 \in \mathbb{R}^{(\mathbb{N})}$ are distinguishable. We say that the system (Σ_{up}) is *observable* if every two distinct points $x^1, x^2 \in \mathbb{R}^{(\mathbb{N})}$ are distinguishable.

Remark 5.2. The system Σ_{up} is observable if and only if for all $x^0 \in \mathbb{R}^{(\mathbb{N})}$ $ce^{tA}x^0 = 0 \Rightarrow x^0 = 0$.

Proposition 5.3. *The system (Σ_{up}) given by (5.1) is observable if and only if*

$$\forall n \in \mathbb{N} \ \text{rank} \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix}_{n|} = n.$$

Proof. Let (Σ_{up}) be not observable. Then there is $v = (v_1, v_2, \dots, v_n, 0, \dots)^T \in \mathbb{R}^{(\mathbb{N})}, v \neq 0$, such that $ce^{tA}v = 0$. From Corollary 2.16: $ce^{tA}v = ce^{tA_{(n)}}v$ and for $k \geq 0 : \frac{d^k}{dt^k}_{|t=0} ce^{tA_{(n)}}v = cA^k_{(n)}v = 0$. As A is upper-diagonal then $cA^k_{(n)} = c_{n|}A^k_{(n)} = (cA^k)_{n|}$. Hence $\begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix}_{n|} v = 0$ so $\text{rank} \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix}_{n|} < n$.

On the other hand, if $n \in \mathbb{N}$ is such that $\text{rank} \begin{pmatrix} c \\ cA \\ \vdots \\ cA^{n-1} \end{pmatrix}_{n|} < n$, then there is $v = (v_1, v_2, \dots, v_n, 0, \dots)^T \in \mathbb{R}^{(\mathbb{N})}, v \neq 0$, such that for every $t \geq 0 : ce^{A_{(n)}t}v =$

0. Since $ce^{tA}v = ce^{tA_{(n)}}v$, the points $x^1 = v$ and $x^2 = 0$ are indistinguishable. Hence (Σ_{up}) is not observable. $\square$

Observe that the conditions in Propositions 5.3 and 4.9 are similar. However the assumptions about the systems are different and the concepts of observability differ as well.

Now we shall consider the transpose A^T of the matrix A. Then
a) A^T is upper-diagonal if A is lower-diagonal,
b) A^T is lower-diagonal and column-finite if A is upper-diagonal and row-finite.

From Propositions 3.13 and 5.3 we have the following.

Proposition 5.4. *Let (Λ) be the system with lower-diagonal matrix A and column-finite b: $\dot{x}(t) = Ax(t) + bu(t),\ x(t) \in \mathbb{R}^{\mathbb{N}}$. The system (Λ) is globally approximately controllable from the origin in finite time if and only if the system:*

$$\begin{aligned} \dot{z}(t) &= A^T z(t), \\ y(t) &= b^T z(t), \quad z(t) \in \mathbb{R}^{(\mathbb{N})} \end{aligned}$$

is observable (on $\mathbb{R}^{(\mathbb{N})}$).

¿From Propositions 3.17 and Corollary 4.7 we have the next proposition.

Proposition 5.5. *Let (Σ) be the system with the upper-diagonal and row-finite matrix A and c being row-finite:*

$$\begin{aligned} \dot{x}(t) &= Ax(t) \\ y(t) &= cx(t). \end{aligned}$$

If the system (Σ) is observable on $\mathbb{R}^{\mathbb{N}}$, then the system:

$$\dot{z}(t) = A^T z(t) + c^T u(t),$$

is globally approximately controllable from the origin in finite time.

The Proposition 5.5 cannot be reversed on the whole space $\mathbb{R}^{\mathbb{N}}$. Indeed the system from Examples 3.18 is globally approximately controllable, but the corresponding system (Σ) from Example 4.8 point b) is not observable on $\mathbb{R}^{\mathbb{N}}$.

Example 5.6. Let the system (Σ) be in the following form

$$\begin{array}{l} \dot{x}_{2n-1} = x_{2n} - x_{2n+2} - x_{2n+3} \\ \dot{x}_{2n} = x_{2n+1} + x_{2n+3} + x_{2n+4} + x_{2n+5} \ , \quad n \in \mathbb{N}. \\ y = x_1 + x_2 + x_3 \end{array}$$

Then the matrix $D = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$ and from Corollary 4.7 (Σ) is observable on $\mathbb{R}^{\mathbb{N}}$.

Let us consider $(\Sigma^T) : \dot{z}(t) = A^T z(t) + bu(t)$, which is described by lower-diagonal and column-finite matrix A^T with the vector $b = c^T$ being a finite column. Then the matrix $\big(b, A^T b, \dots\big) = D^T$ and (Σ^T) is globally approximately controllable in finite time.

References

[1] S. Banach, *Théorie des opérations linéaires*, Warsaw 1932.

[2] Z. Bartosiewicz, D. Mozyrska, Observability of infinite–dimensional finitely presented discrete–time linear systems, *Zeszyty Naukowe Politechniki Białostockiej. Matematyka-Fizyka-Chemia* 20 (2001), 5–14.

[3] Z. Bartosiewicz, D. Mozyrska, Observability of row–finite countable systems of linear differential equations, Proceedings of 16th IFAC Congress, 4-8 July 2005, Prague

[4] R.G. Cooke, *Infinite Matrices and Sequence Spaces*, London 1950

[5] R.F. Curtain, A.J. Pritchard, *Infinite Dimensional Linear Systems Theory*, Springer-Verlag, Berlin 1978.

[6] R. Deimling, *Ordinary Differential Equations in Banach Spaces*, Lecture Notes in Mathematics, vol. 596, Springer-Verlag 1977.

[7] R.E. Edwards, *Functional analysis: theory and applications*, Holt, Rinehart & Winston, 1965

[8] M. Fliess *et al.*, On nonlinear controllability, infinite jets and prolongations, in: Proceedings of European Control Conference ECC-97, Brussels, Belgium, July 1997.

[9] G. Herzog, On Lipschitz conditions for ordinary differential equations in Fréchet spaces, Czech. Math. J. 48 (1998), 95–103.

[10] B. Jakubczyk, Remarks on equivalence and linearization of nonlinear systems, in: Proc. Nonlinear Control Systems Design Symposium, Bordeaux, France, 1992.

[11] L.V. Kantorovich, G.P. Akilov, *Functional Analysis*, Pergamon Press Ltd, Second Edition 1982.

[12] J.L. Kelley, *General topology*, v. Nostrand, 1955

[13] J. Klamka, *Controllability of Dynamical Systems*, PWN-Polish Scientific Publishers in co-edition with Kluwer Academic Publishers, Warsaw, 1991.

[14] K. Kowalski, W-H. Steeb, *Nonlinear Dynamical Systems and Carleman Linearization*, World Scientific Publishing Co. Pte. Ltd., Singapore, 1991.

[15] R. Lemmert, On ordinary differential equations in locally convex spaces, *Nonlinear Anal.* 10 (1986), 1385–1390.

[16] K. Moszyński, A. Pokrzywa, Sur les systémes infinis d'équations différentielles ordinaires dans certain espaces de Fréchet, *Dissert. Math.* 115 (1974).

[17] D. Mozyrska, Z. Bartosiewicz, Local observability of systems on $\mathbb{R}^\infty$, in: Proceedings of MTNS'2000, Perpignan, France, 2000.

[18] D. Mozyrska, Local observability of infinitely-dimensional finitely presented dynamical systems with output (in Polish), Ph.D. thesis, Technical University of Warsaw, Poland, 2000.

[19] K.P. Persidski, Countable system of differential equations and stability of their solutions (in Russian), Izv. Akad. Nauk Kazach. SSR 7 (1959) 52–71.

[20] J.B. Pomet, A differential geometric setting for dynamic equivalence and dynamic linearization, in: Banach Center Publications 32 (1995), 319-339.

[21] S. Rolewicz, *Functional Analysis and Control Theory, Linear Systems*, PWN-Polish Scientific Publishers in co-edition with D. Reidel Publishing Company, Warsaw, 1977.

[22] J. Ruiz, *The Basic Theory of Power Series*, Vieweg, 1993.

[23] A.E. Taylor, D.C. Lay, *Introduction to Functional Analysis,* Wiley, 1980.

[24] R. Triggiani, On the lack of exact controllablity for mild solutions in Banach spaces, *Journal of Mathematical Analysis and Applications* 50 (1975), 438-446.

[25] A. Wilansky, K. Zeller, Inverses of matrices and matrix transformations, *Proc. American Math. Soc.* 30 (1955) 123–7.

[26] O.A. Zautykov, Countable systems of differential equations and their applications (in Russian), *Diff. Uravn.* 1 (1965) 162–170.

[27] O.A. Zautykov, K.G. Valeev, *Infinite systems of differential equations* (in Russian), Izdat. "Nauka" Kazach. SSR, Alma-Ata 1974.