Unification of continuous-time and discrete-time systems: the linear case*

Zbigniew Bartosiewicz and Ewa Pawłuszewicz

Białystok Technical University
Wiejska 45a, Bialystok, Poland
e-mail: bartos@pb.bialystok.pl; epaw@cksr.ac.bialystok.pl

1 Introduction

Control theory splits into two parallel areas: in the first one time is continuous, in the second one – discrete. The statements in both theories are usually similar or even identical. This is true for linear as well for nonlinear systems. The language of time scales, created some 25 years ago by Stefan Hilger [2], seems to be an ideal tool to unify the two theories. One of the main concepts is the delta derivative, which is a generalization of ordinary (time) derivative. If the time scale is the real line, we get ordinary derivative. In the case of integer numbers, delta derivative of a function is the difference of its values at subsequent points. Thus differential as well difference equations are included in the theory. But one can also consider mixed cases when time is partly discrete and partly continuous. We recall basic definitions and facts. More information can be found e.g. in [1].

We develop here the basics of linear control theory on time scales. In particular we present criteria of controllability and observability for systems on time scales. Formally, they are extensions of well known facts in both discrete-time and continuous-time theories.

2 Delta derivative

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases comprise $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$, $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, but $\mathbb{T}$ can also be the Cantor set or $\mathbb{T} = \bigcup_{k=0}^{\infty} [k(a+b), k(a+b)+a]$ for $a, b > 0$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. For $t \in \mathbb{T}$ we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

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If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Points that are right- and left-scattered at the same time are called *isolated*. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*; if $t > \inf \mathbb{T}$ and $\rho(t) = t$ then t is called *left-dense*. Points that are right- and left-dense at the same time are called *dense*.

Finally we define the set

$$\mathbb{T}^k := \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}] & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases}$$

Example 2.1.

- If $\mathbb{T} = \mathbb{R}$ then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
- If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.

Definition 2.2. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^k$. *Delta derivative* of f at t, denoted by $f^{\triangle}(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^{\triangle}(t)(\sigma(t) - s)| \leq \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}^k$.

If $f : \mathbb{T} \rightarrow \mathbb{C}$, then we set $f^{\triangle} := (\operatorname{Re} f)^{\triangle} + i(\operatorname{Im} f)^{\triangle}$.

It can be noticed that in the general case forward jump operator σ is not delta differentiable.

Remark 2.3.

- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \rightarrow \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ iff

  $$f^{\triangle}(t) = \lim_{s \longrightarrow t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

  i.e. iff f is differentiable in the ordinary sense at t.

- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \rightarrow \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with

  $$f^{\triangle}(t) = \frac{f(\delta(t)) - f(t)}{\mu(t)} = f(t + 1) - f(t) = \triangle f(t)$$

  where $\triangle$ is the usual forward difference operator defined by the last equation above.

Example 2.4. The delta derivative of t^2 is $t + \sigma(t)$. This means that the second delta derivative of t^2 may not exists.

The delta-derivative of $\frac{1}{t}$ is $\frac{-1}{t\sigma(t)}$.

3 Antiderivative

A function $f : \mathbb{T} \to \mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points at $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. It can be shown that
f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated
and that σ is rd-continuous.

The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are n times delta differentiable and whose all derivatives are rd-continuous will be called of the C^n *rd-class*.

A continuous function $f : \mathbb{T} \to \mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D, provided $D \subset \mathbb{T}^k$, $D \setminus \mathbb{T}^k$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t \in D$. It can be proved that if f is regulated then there exists a function F that is pre-differentiable with region of differentiation D such that

$$F^{\triangle}(t) = f(t)$$

for all $t \in D$. Any such function is called pre-antiderivative of f. Then *indefinite integral* of f is defined by

$$\int f(t) \triangle t = F(t) + C$$

where C is an arbitrary constant. *Cauchy integral* is

$$\int_r^s f(t) \triangle t = F(s) - F(r) \quad \text{for all} \quad r, s \in \mathbb{T}^k$$

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\triangle}(t) = f(t)$ holds for all $t \in \mathbb{T}^k$.

Remark 3.1. It can be shown that every rd-continuous function has an antiderivative. Moreover, if $f(t) \geq 0$ for all $a \leq t < b$ and $\int_a^b f(\tau) \triangle \tau = 0$ then $f \equiv 0$.

Example 3.2.

- If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(\tau) \triangle \tau = \int_a^b f(\tau) d\tau$, where the integral on the right is the usual Riemann integral.
- If $\mathbb{T} = \mathbb{Z}$, then $\int_a^b f(\tau) \triangle \tau = \sum_{t=a}^{b-1} f(t)$ for $a < b$.
- If $\mathbb{T} = h\mathbb{Z}$, $h > 0$, then $\int_a^b f(\tau) \triangle \tau = \sum_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a < b$.

Remark 3.3. An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula of an antiderivative of t: antiderivative of $\frac{t^2}{2}$ is $\frac{t+\sigma(t)}{2} = t + \frac{\mu(t)}{2}$.

Example 3.4. If $\mathbb{T} = \mathbb{Z}$ and $a \neq 1$, then $\int a^t \triangle t = \frac{a^t}{a-1} + C$, since $(\frac{a^t}{a-1})^{\triangle} = \frac{a^{t+1}-a^t}{a-1} = a^t$.

4 Linear differential systems

Let A be an $m \times n$ matrix-valued function on $\mathbb{T}$. A is *rd-continuous* on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$. The set of all such matrices is denoted by $C_{rd} = C_{rd}(\mathbb{T}, \mathbb{R}^{m\times n})$. Occasionally A will have complex components.

We say that A is differentiable on $\mathbb{T}$, provided each entry of A is differentiable on $\mathbb{T}$. In this case we put $A^{\triangle} = (a_{ij}^{\triangle})_{1\leq i\leq m;1\leq j\leq n}$. By A^* we will denote a (conjugate) transpose of A. It can be shown that $(A^*)^{\triangle} = (A^{\triangle})^*$.

An $n \times n$ matrix-valued function A on $\mathbb{T}$ is called *regressive* with respect to $\mathbb{T}$ provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$ (I denotes the identity matrix). The system of delta differential equations $x^{\triangle} = A(t)x$ is called *regressive* provided A is regressive.

Remark 4.1.

- If $\mathbb{T} = \mathbb{R}$, then any matrix-valued function A on $\mathbb{T}$ satisfies the regressivity condition. Moreover, in this case A is rd-continuous iff it is continuous.

- If $\mathbb{T} = \mathbb{Z}$, then any matrix-valued function A on $\mathbb{T}$ is rd-continuous, since there are no left- or right-dense points. In order that A be regressive, the matrix $I + A(t)$ needs to be invertible for each $t \in \mathbb{Z}$, which holds iff $A(t)$ has no eigenvalue equal to -1 for each $t \in \mathbb{Z}$. In this case the equation $x^{\triangle} = A(t)x$ is equivalent to $x(t+1) = (I + A(t))x(t)$. Regressivity means that the last equation can be solved backwards.

Theorem 4.2. *[1] Let A be regressive and rd-continuous $n \times n$ matrix-valued function on $\mathbb{T}$. Then the initial value problem*

$$x^{\triangle} = A(t)x, \quad x(t_0) = x_0$$

has a unique solution x defined on $\mathbb{T}$.

Let $t_0 \in \mathbb{T}$ and let A be regressive and rd-continuous $n \times n$ matrix-valued function. The unique matrix-valued solution of the initial value problem $X^{\triangle} = A(t)X$, $X(t_0) = I$, is called the *matrix exponential function of* A (at t_0). Its value at $t \in \mathbb{T}$ will be denoted by $e_A(t, t_0)$

Remark 4.3. Let A be a constant $n \times n$ matrix.

- If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$.

- If $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then $e_A(t, t_0) = (I + A)^{(t-t_0)}$.

It can be proved that

1. $e_0(t,s) = I$ and $e_A(t,t) = I$ for every $t, s \in \mathbb{T}$;
2. $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t,s)$;
3. $e_A(t,s) = e_A^{-1}(s,t)$;
4. $e_A(t,s)e_A(s,r) = e_A(t,r)$.

Let us consider the nonhomogeneous equation

$$x^{\triangle} = A(t)x + f(t), \quad x(t_0) = x_0 \tag{1}$$

where $f : \mathbb{T} \to \mathbb{R}^n$ is a vector-valued rd-continuous function, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$.

Theorem 4.4. *Let A be a rd-continuous regressive $n \times n$ matrix-valued function on $\mathbb{T}$. Then the initial value problem (1) has a unique solution on $\mathbb{T}$, given by*

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau)) f(\tau) \triangle\tau \tag{2}$$

5 Control systems

Let us consider a linear time-variant control system with outputs

$$\begin{aligned} x^{\triangle}(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned} \tag{3}$$

with the initial condition $x(t_0) = x_0$. Suppose that $t \in \mathbb{T}$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ and $A(t), B(t), C(t), D(t)$ are matrix-valued functions on $\mathbb{T}$ of dimensions respectively $n \times n$, $n \times m$, $p \times n$ and $p \times m$. We assume that A is regressive.

We say that system (3) is *controllable* if any state $x \in \mathbb{R}^n$ can be reached from any other state in finite time using piecewise constant controls.

Two states are *indistinguishable* if trajectories starting at those points and corresponding to the same control give rise to the same output for all nonnegative times. The systems is *observable* if it does not have distinct indistinguishable points.

The following facts are generalizations of well known standard results on controllability and observability (see e.g. [3, 4]). They correspond directly to similar results for the continuous-time case. However, the discrete time case is more involved, as the delta derivative in this case gives the difference operator and not the time shift, which is usually used.

Let us consider the matrix

$$W(t_0, t_k) = \int_{t_0}^{t_k} e_A(t_k, \sigma(\tau)) B(\tau) B^*(\tau) e_A^*(t_k, \sigma(\tau)) \triangle\tau \tag{4}$$

One can notice that $W^*(t_0,t_k) = W(t_0,t_k)$ and $W(t_0,t_k) \geq 0$.

Theorem 5.1. *Let us assume that the matrix (4) is nonsingular. For any* $x_0, x_k \in \mathbb{R}^n$ *the control*

$$u(t) = B^*(t)e_A^*(t_k, \sigma(t))W^{-1}(t_0, t_k)[x_k - e_A(t_k, t_0)x_0] \quad (5)$$

gives $x(t_k, u_{|(t_0,t_k]\cap\mathbb{T}}, x_0) = x_k$*, i.e.* x_k *is the solution of the first equation of (3) evaluated at time* t_k*.*

Theorem 5.2. *The system (3) is controllable if and only if for some finite* $t_0, t_k \in \mathbb{T}$*,* $t_k > t_0$*, the matrix* $W(t_0, t_k)$ *is nonsingular.*

Now let us consider the matrix

$$M(t_0, t_k) = \int_{t_0}^{t_k} e_A^*(\tau, t_0)C^*(\tau)C(\tau)e_A(\tau, t_0)\triangle\tau \quad (6)$$

One can see that $M^*(t_0, t_k) = M(t_0, t_k)$ and $M(t_0, t_k) \geq 0$.

Theorem 5.3. *The system (3) is observable if and only if for some finite* $t_0, t_k \in \mathbb{T}$*,* $t_k > t_0$*, matrix* $M(t_0, t_k)$ *is nonsingular.*

6 Conclusion

We presented here extension of controllability and observability results to systems on arbitrary time scales. We studied only linear systems, so the whole area of nonlinear control is waiting for a lift to arbitrary time scales. Moreover, we have assumed that the matrix of coefficients of the linear system is regressive, which for discrete-time systems implies existence of backward solutions. Definitely this is a restrictive assumption and one may try to relax it. This would, however, require developing a new theory of linear differential equations on time scales.

References

[1] M. Bohner, A. Peterson, Dynamic Equations on Time Scales, Birkhauser 2001.

[2] S. Hilger, Ein Maßkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten, Ph.D. thesis, Universität Würzburg, 1988.

[3] E.B. Lee, L. Markus, *Foundations of Optimal Control Theory*, Krieger Publishing Company 1986.

[4] E.D.Sontag, *Mathematical Control Theory*, Springer-Verlag 1990