Realizations of Nonlinear Control Systems on Time Scales

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***Abstract*— Nonlinear partially defined systems on an arbitrary unbounded time scale are studied. They include continuous-time and discrete-time systems. The main problem is to find necessary and sufficient conditions for an abstract input/output map to have a realization as a nonlinear system of a specific class on the time scale. The obtained results extend criteria of realizability of continuous-time polynomial systems. A simple construction of a realization is provided.**

I. Introduction

A dynamical control system with an output, initialized at some initial state, gives rise to the response map, which assigns to each control (input) u defined on the interval $[T_0^u, T_1^u]$ the value of the output at time T_1^u. A realization of an abstract response map consists of a dynamical system with an output, initialized at some initial state, whose response map coincides with the abstract response map. In such a setting the realization problem was studied e.g. by E. Sontag [1], B. Jakubczyk [2] and Z. Bartosiewicz [3]. The systems were either polynomial or analytic, with discrete or continuous time. Besides the response map, other input/output representations are widely used. They include the input/output map, the Volterra or Fliess series (see e.g. [4]), input/output relations like differential or difference equations of higher order (see e.g. [5]). We refer the reader to overviews [6] and [7], where different approaches are compared and more references can be found.

A time scale is a model of time. Time may be continuous or discrete, or partly continuous and partly discrete. Theory of systems on time scales allows for unified treatment of discrete-time and continuous-time systems, and systems where time is mixed. Calculus on time scales was originated in 1988 by S. Hilger [8]. It includes differential and integral calculus, and allows for studying differential and difference equations in a common setting. In Section II we give the basic facts on this calculus and refer to [9] for more details and more references.

Though the literature on dynamical equations on time scales is abundant, control theory on time scales is not much developed. Mainly linear systems and basic properties were studied (see [10]–[13]). Realization theory of linear systems was presented in [13] (time-variant systems) and [12] (time-invariant systems). See [14] for another attempt to unify discrete-time and continuous-time theories.

In this paper we study a realization problem for

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nonlinear control systems with output, whose dynamics are defined by differential equations on time scales and that are described by functions of class C^k, where k may be a natural number, or $k = \infty$ (smooth systems), or $k = \omega$ (analytic systems), or $k = \mathrm{pol}$ (polynomial systems). We consider piecewise constant controls defined on subsets of a time scale $\mathbb{T}$ and a response map P that is to be realized. The main result gives a necessary and sufficient condition for P to have a realization of class C^k. It is an extension of a criterion in [3], where polynomial continuous-time systems were studied, but it is expressed in a simpler and more direct language. On the other hand we lose some structural relations that were present in [3], like homomorphism between the observation algebra of the system and the observation algebra of its response map, or relations between differential operators acting on different levels. The operators are still present, but they are no longer derivations of algebras. As we study systems that are not necessarily polynomial, algebras used in [3] are not useful now. Addition and multiplication are replaced by substitutions into functions of class C^k on $\mathbb{R}^n$. One could follow this idea introducing function universes and universe spaces related to systems and response maps (see [15]–[17]). However, at the moment, it seems that this language will not give enough profit to the solution of the main problem to justify the burden of explaining its ideas.

The criterion of realizability is simple, but not constructive. We have to find a finite number of functions that satisfy certain conditions. There is no algorithm to produce them and the same situation was in [3]. However the example provided in the paper shows how to search for the functions.

II. Preliminaries

We give here a short introduction to differential calculus on time scales and set necessary notation. More material on time scales can be found in [9].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. In this paper we shall assume that $\sup \mathbb{T} = +\infty$. The time scale $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. For $\mathbb{T}$ we define

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ by $\mu(t) := \sigma(t) - t$.

Example 2.1:

- If $\mathbb{T} = \mathbb{R}$ then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$; the graininess function $\mu(t) \equiv 0$.
- If $\mathbb{T} = \mathbb{Z}$ then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$; the graininess function $\mu(t) \equiv 1$.
- Let $q > 1$ and $\mathbb{T} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$. Then $\sigma(t) = qt$, $\rho(t) = \frac{t}{q}$ and $\mu(t) = (q - 1)t$ for all $t \in \mathbb{T}$.

If $a, b \in \mathbb{T}$, then $[a, b] := \{t \in \mathbb{T} : a \le t \le b\}$. Similarly for (a, b) and half-open intervals.

Definition 2.2: Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$. The *delta derivative* of f at t, denoted by $f^{\triangle}(t)$ (or by $\frac{\Delta}{\Delta t} f(t)$), is the real number (provided it exists) with the property that given any ε there is a neighborhood $U = (t - \delta, t + \delta)$ (for some $\delta > 0$) such that

$$|(f(\sigma(t)) - f(s)) - f^{\triangle}(t)(\sigma(t) - s)| \le \varepsilon |\sigma(t) - s|$$

for all $s \in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}$ provided $f^{\triangle}(t)$ exists for all $t \in \mathbb{T}$.

Remark 2.3: We shall often drop the word 'delta' and say that f is differentiable on $\mathbb{T}$. If in Definition 2.2 we change the neighborhood U for a one-sided neighborhood, we get the definition of one-sided derivative.

Example 2.4:

- If $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{R}$ if and only if f is differentiable in the ordinary sense at t. Then $f^{\triangle}(t) = \lim_{s \longrightarrow t} \frac{f(t)-f(s)}{t-s} = f'(t)$.
- If $\mathbb{T} = \mathbb{Z}$, then $f : \mathbb{Z} \to \mathbb{R}$ is always delta differentiable at every $t \in \mathbb{Z}$ with $f^{\triangle}(t) = f(t+1) - f(t)$.
- If $\mathbb{T} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$ and $q > 1$, then $f^{\triangle}(t) = \frac{f(qt)-f(t)}{(q-1)t}$ for all $t \in \mathbb{T} \setminus \{0\}$.

Theorem 2.5: [9] Let $x : \mathbb{T} \to \mathbb{R}^n$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then

$$\frac{\Delta}{\Delta t}\varphi(x(t)) = \int_0^1 [\varphi'(x(t) + h\mu(t)x^{\triangle}(t)]dh \cdot x^{\triangle}(t) \quad (1)$$

where φ' denotes the (standard) gradient of φ.

Let $f : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. Consider the (delta) differential equation

$$x^{\Delta}(t) = f(t, x(t)). \quad (2)$$

A *solution* to (2) is a function x defined on some interval $[a, b] \subseteq \mathbb{T}$ and satisfying (2). If f is continuous with respect to t (the first variable) and of class C^1 with respect to x (the second variable), then for every initial condition $x(t_0) = x_0$ there exists a unique forward solution defined of some interval $[t_0, t_1]$. It may happen, however, that this solution cannot be extended backwards.

Consider the linear scalar equation

$$x^{\Delta}(t) = a(t)x(t), \quad (3)$$

where $a(t)$ and $x(t)$ belong to $\mathbb{R}$ for $t \in \mathbb{T}$ and a is continuous. The forward solution of (3) for the initial condition $x(t_0) = 1$ is called the *exponential function of a centered at t_0*. It is defined for all $t \geq t_0$ and its value at t is denoted by $e_a(t, t_0)$. We shall need the following property

$$e_a(t, t_0) = e_a(t, s)e_a(s, t_0), \quad (4)$$

which holds for any $t_0 \leq s \leq t$.

Let D be an open subset of $\mathbb{R}^n$ and $F : D \to \mathbb{R}^m$. Then F is called a *partially defined function* on $\mathbb{R}^n$, with the domain D denoted by $\mathrm{dom}F$. We shall consider partially defined functions on $\mathbb{R}^n$ of class C^k, where $k \in \mathbb{N}$, or $k = \infty$ (smooth functions), or $k = \omega$ (real analytic functions), or $k = \mathrm{pol}$ (polynomials). If A is any set and $\varphi_i : A \to \mathbb{R}$ for $i = 1, \ldots, n$, then $F(\varphi_1, \ldots, \varphi_n)$ denotes the substitution of $\varphi_1, \ldots, \varphi_n$ into F

$$F(\varphi_1, \ldots, \varphi_n)(a) := F(\varphi_1(a), \ldots, \varphi_n(a))$$

defined for all $a \in A$ such that $(\varphi_1(a), \ldots, \varphi_n(a)) \in \mathrm{dom}F$. Let $C^k(\varphi_1, \ldots, \varphi_n)$ denote the set of all substitutions $F(\varphi_1, \ldots, \varphi_n)$ defined for every $a \in A$, where F is a partially defined C^k function on $\mathbb{R}^n$ with values in $\mathbb{R}$.

III. Problem statement

Let Ω be an arbitrary set. It will be called the *set of control values*. Let $\omega \in \Omega$ and $t_0, t_1 \in \mathbb{T}$, $t_0 \leq t_1$. A function $u : [t_0, t_1] \to \Omega$ defined by $u(t) = \omega$ is called a *constant control* and denoted by $[\omega, t_0, t_1]$. If $t_0 = t_1$, u is called an *empty control at* t_0 and denoted by $\emptyset_{t_0}$. Thus $\emptyset_{t_0} = [\omega, t_0, t_0]$ for some $\omega \in \Omega$. An *empty control* is $\emptyset_{t_0}$ for some $t_0 \in \mathbb{T}$. If $t_0 < t_1$, the constant control is called *nonempty*. For $u = [\omega, t_0, t_1]$ we define $T_0^u := t_0$ and $T_1^u := t_1$.

Let us consider constant nonempty controls u and v for which $T_0^v = T_1^u$. The *concatenation* of controls u and v, denoted by $u \sqcup v$, is the multivalued function w

defined on $[T_0^u, T_1^v]$ by

$$w(t) = \begin{cases} u(t), & t \in [T_0^u, T_1^u) \\ \{u(T_1^u), v(T_0^v)\}, & t = T_0^v = T_1^u \\ v(t), & t \in (T_0^v, T_1^v] \end{cases}$$

Observe that the value of w at T_1^u is a subset of Ω consisting of two elements (if $u(T_1^u)$ and $v(T_0^v)$ are distinct). We assume that one can recover u (the earlier control) and v (the later control) from their concatenation $u \sqcup v$. This means that if $w(t) = \{\omega_1, \omega_2\}$, then one can decide whether $u(t) = \omega_1$ or $u(t) = \omega_2$. We call T_1^u a *switching time* of w. All other instants for which w is defined are called *nonswitching*. We set $T_0^w = T_0^u$, $T_1^w = T_1^v$, and use the above formula to define recurrently *piecewise constant controls* as concatenations of a finite number of nonempty constant controls. By *control* we mean a piecewise constant (in particular, constant) control or empty control. The set of all controls will be denoted by U.

Remark 3.1: Observe that for a piecewise constant control u any $t \in (T_0^u, T_1^u)$ may be a switching time and $u(t)$ may have two values. This may look strange, but such construction allows for unified treatment of controls and control systems defined on different time scales.

Let $w \in U$ and $t \in [T_0^w, T_1^w]$. By $w_{|[T_0^w,t]}$ we denote w restricted to the interval $[T_0^w, t]$. This has a standard meaning besides, possibly, its value at t. If t is a switching time of w, i.e. $w = u \sqcup v$ and $T_1^u = t$, then we set $w_{|[T_0^w,t]}(t) = u(t)$. In this way we obtain again a control belonging to U.

For each $\omega \in \Omega$ let $f_\omega : \mathrm{dom}(f_\omega) \to \mathbb{R}^n$ be a partially defined C^k map on $\mathbb{R}^n$, so its domain is an open subset of $\mathbb{R}^n$. Similarly, let $h : \mathrm{dom}(h) \to \mathbb{R}^r$ be a partially defined C^k map on $\mathbb{R}^n$. We shall assume that the set $\mathrm{dom}(h) \cap (\bigcap_{\omega\in\Omega} \mathrm{dom}(f_\omega))$ is nonempty and contains a subset X, which will be called the *state space*. Let us consider a nonlinear control system with output, denoted by Σ:

$$x^{\triangle}(t) = f_{u(t)}(x(t)) \tag{5a}$$

$$y(t) = h(x(t)) \tag{5b}$$

where $t \in \mathbb{T}$, $u(t) \in \Omega$, $x(t) \in X$ and $y(t) \in \mathbb{R}^r$.

Let us choose an initial point $x_0 \in X$ and a control u. The *trajectory* of Σ from x_0 corresponding to the control u is a function $x = \psi(\cdot, x_0, u) : [T_u^0, T_u^1] \to X$ defined as follows. If $u = \emptyset_{t_0}$, then x is defined only at t_0 and $x(t_0) = \psi(t_0, x_0, u) = x_0$. If $u = [\omega, t_0, t_1]$, then x is the unique solution to the initial value problem:

$$x^{\triangle}(t) = f_\omega(x(t)), \quad x(t_0) = x_0,$$

provided it is defined on the interval $[t_0, t_1]$ and for all $t \in [t_0, t_1]$, $x(t) \in X$. Observe that such a trajectory may not exist. If $u = [\omega_1, t_0, t_1] \sqcup [\omega_2, t_1, t_2]$, then x restricted to $[t_0, t_1]$ is the trajectory from x_0 corresponding to the constant control $u_1 = [\omega_1, t_0, t_1]$. To extend x to the entire interval $[t_0, t_2]$, we solve the initial value problem

$$x^{\triangle}(t) = f_{\omega_2}(x(t)), \quad x(t_1) = \psi(t_1, x_0, u_1).$$

For controls that are concatenations of more than two constant controls, the procedure is similar, but, as before, there is no guarantee that the trajectory exists.

A control u is called *admissible* for $x_0 \in \mathbb{R}^n$ if there exists a trajectory of Σ from x_0 corresponding to a control u. If it exists, such a trajectory is unique. The set of all controls admissible for x_0 will be denoted by U_{Σ,x_0}.

Observe that the set U_{Σ,x_0} satisfies the following conditions:

A. For every $t_0 \in \mathbb{T}$, $\emptyset_{t_0} \in U_{\Sigma,x_0}$.

B. For every $u \in U_{\Sigma,x_0}$ and $t \in [T_0^u, T_1^u]$, the control $u_{|[T_0^u,t]} \in U_{\Sigma,x_0}$.

We shall assume that additionally the following holds:
C. For every $u \in U_{\Sigma,x_0}$ and every $\omega \in \Omega$, there exists $\varepsilon > 0$ such that $T_1^u + \varepsilon \in \mathbb{T}$ and $u \sqcup [\omega, T_1^u, T_1^u + \varepsilon] \in U_{\Sigma,x_0}$.

Remark 3.2: The condition **C** means a kind of local invariance of the state space X with respect to the dynamics (5a) of the system Σ. If $X = \mathbb{R}^n$, the condition C always holds. Moreover, if the time scale $\mathbb{T}$ is discrete, e.g. $\mathbb{T} = \mathbb{Z}$, **C** means that $U_{\Sigma,x_0} = U$.

For the system Σ and the initial point x_0 we define the *response map*:

$$P_{\Sigma,x_0} : U_{\Sigma,x_0} \to \mathbb{R}^r : u \mapsto y(T_1^u) := h(\psi(T_1^u, x_0, u)). \tag{6}$$

Now let us consider a map $P : U_P \to \mathbb{R}^r$, where U_P is a subset of U. The realization problem can be stated now as follows:
When there exists a C^k system Σ and an initial point x_0 such that $U_P \subset U_{\Sigma,x_0}$ and $P(u) = P_{\Sigma,x_0}(u)$ for all $u \in U_P$? If such Σ and x_0 exist, how to construct them?

IV. Realizations of response maps

Let φ be a partially defined real C^k function on $\mathbb{R}^n$, D be an open subset of $\mathbb{R}^n$, $f : D \to \mathbb{R}^n$ be of class C^k and $t_0 \in \mathbb{T}$. Let us define the operator $\Gamma_f^{t_0}$ by

$$(\Gamma_f^{t_0}\varphi)(x) := \int_0^1 \varphi'(x + h\mu(t_0)f(x))dh \cdot f(x), \tag{7}$$

where φ' is the gradient of φ. Thus $\Gamma_f^{t_0}\varphi$ is again a partially defined C^k function on $\mathbb{R}^n$, but its domain may be smaller than the domain of φ (even empty).

It can be noticed that if $\mu(t_0) \neq 0$ then

$$\begin{aligned}(\Gamma_f^{t_0}\varphi)(x) &= \frac{1}{\mu(t_0)} \int_0^1 \frac{d}{dh}(\varphi(x + h\mu(t_0)f(x))dh \\ &= \frac{1}{\mu(t_0)}(\varphi(x + \mu(t_0)f(x)) - \varphi(x))\end{aligned}$$

For $\mu(t_0) = 0$ we obtain $(\Gamma_f^{t_0}\varphi)(x) = \varphi'(x)f(x)$. As f may be interpreted as a vector field on $D \subset \mathbb{R}^n$, $(\Gamma_f^{t_0}\varphi)(x)$ is then equal $L_f\varphi$ – the Lie derivative of the function φ with respect to the vector field f. In general, when operator $\Gamma_f^{t_0}\varphi$ does not depend on t_0, we will denote it by $\Gamma_f\varphi$.

Example 4.1:

- If $\mathbb{T} = \mathbb{R}$, then $\Gamma_f = L_f$.
- If $\mathbb{T} = \mathbb{Z}$, then $(\Gamma_f\varphi)(x) = \varphi(x + f(x)) - \varphi(x)$.
- If $\mathbb{T} = q^{\mathbb{N}}$, $q > 1$, then $(\Gamma_f^{t_0}\varphi)(x) = \frac{\varphi(x+t_0(q-1)f(x))-\varphi(x)}{(q-1)t_0}$.

Example 4.2: Let $\varphi = x^i$ be the i-th coordinate function on $\mathbb{R}^n$. Then $(x^i)' = e_i$ – the vector of the standard basis of $\mathbb{R}^n$ with 1 at the i-th position. For any $t_0 \in \mathbb{T}$ we have

$$(\Gamma_f^{t_0} x^i)(x) = e_i f(x) = f_i(x).$$

Let $P : U_P \to \mathbb{R}^r$ be an abstract response map that is to be realized as the response map of a C^k system on $\mathbb{R}^n$ (for some n) and an initial condition. To achieve this we impose some preliminary assumptions on P and U_P. First we assume that U_P satisfies conditions **A**, **B** and **C** from Section III. We shall be studying real functions on U_P. We assume that any such function is constant on the set of all empty controls. This concerns, in particular, the components P_i, $i = 1, \dots, r$, of the map P.

Let $\psi : U_P \to \mathbb{R}$ and let $\omega \in \Omega$. Define the operator Δ_ω in the following way:

$$(\Delta_\omega\psi)(u) := \frac{\Delta}{\Delta t}\Big|_{t=T_1^u+} \psi(u \sqcup [\omega, T_1^u, t]) \tag{8}$$

for any $u \in U_P$ and $t \geq T_1^u$. Observe that in (8) we compute the right-hand derivative at T_1^u. Let $\mathcal{A}_P$ denote the set of all functions $\psi : U_P \to \mathbb{R}$ such that $(\Delta_\omega\psi)(u)$ is defined for all $\omega \in \Omega$ and all $u \in U_P$, and, moreover the map

$$t \mapsto \psi(u_{|[T_0^U,t]})$$

is differentiable for any nonempty $u \in U_P$ at all points t that are not switching points of u. We assume that all the components of P belong to $\mathcal{A}_P$.

Remark 4.3: The operators Δ_ω were earlier used in [3] and [2] for the continuous-time case. In [3] the observation algebra of P was defined as the smallest algebra of real functions on U_P that contains P_i, $i = 1, \ldots, r$, and is stable under the action of Δ_ω, $\omega \in \Omega$. This algebra was used to characterize conditions under which P may be realized as the response map of a polynomial system on $\mathbb{R}^n$. In [2] the operators Δ_ω were used to express a rank condition that was equivalent to existence of an analytic realization.

Let us consider now the system Σ given by (5) and the initial state x_0.

Lemma 4.4: Let $\xi : U_{\Sigma,x_0} \to \mathbb{R}^n$ be the map defined by $\xi(u) := \psi(T_1^u, x_0, u)$. Then for any partially defined C^k function φ on $\mathbb{R}^n$, any $\omega \in \Omega$ and any $u \in U_{\Sigma,x_0}$

$$[\Delta_\omega(\varphi \circ \xi)](u) = [(\Gamma_{f_\omega}^{T_1^u} \varphi) \circ \xi](u).$$

Proof: Let us fix $\omega \in \Omega$ and for $t > T_1^u$, $t \in \mathbb{T}$, let $w_t = u \sqcup [\omega, T_1^u, t]$. Then we have

$$\begin{aligned}
&[\Delta_\omega(\varphi \circ \xi)](u) = \frac{\Delta}{\Delta t} \,|_{t=T_1^u+}\, \varphi(\xi(w_t)) \\
&= \frac{\Delta}{\Delta t} \,|_{t=T_1^u+}\, \varphi(\psi(t, x_0, w_t)) \\
&= \int_0^1 \varphi'(\psi(T_1^u, x_0, u) + h\mu(T_1^u) f_\omega(\psi(T_1^u, x_0, u)))dh \\
&\qquad \cdot f_\omega(\psi(T_1^u, x_0, u)) \\
&= (\Gamma_{f_\omega}^{T_1^u} \varphi)(\psi(T_1^u, x_0, u)) \\
&= [(\Gamma_{f_\omega}^{T_1^u} \varphi) \circ \xi](u)
\end{aligned}$$

■

Remark 4.5: The map ξ that appears in Lemma 4.4 is the reachability map of Σ. For a fixed initial condition x_0 it assigns to each admissible control u the state reached at the time T_1^u. A special version of Lemma 4.4 was used in [3] to construct a homomorphism between the observation algebra of the system and the observation algebra of its response map.

We present now the main result of this paper.

Theorem 4.6: The map $P : U_P \to \mathbb{R}^r$ has a C^k realization if and only if there exist $\varphi_1, \ldots, \varphi_n \in \mathcal{A}_P$ such that for all $\omega \in \Omega$, $\Delta_\omega(\varphi_i) \in C^k(\varphi_1, \ldots, \varphi_n)$ and for all $i = 1, \ldots, r$, $P_i \in C^k(\varphi_1, \ldots, \varphi_n)$.

Proof: "$\Leftarrow$" Let $\Phi = (\varphi_1, \ldots, \varphi_n)$. We shall construct a realization of the map P on $X := \Phi(U_P) \subseteq \mathbb{R}^n$. The condition $\Delta_\omega(\varphi_i) \in C^k(\varphi_1, \ldots, \varphi_n)$ implies that $\Delta_\omega(\varphi_i) = f_{\omega,i}(\varphi_1, \ldots, \varphi_n)$, where $f_{\omega,i}$ is a partially defined C^k function of n variables. Let us define $f_\omega := (f_{\omega,1}, \ldots, f_{\omega,n})$. From the fact that $P_j \in C^k(\varphi_1, \ldots, \varphi_n)$ for $j = 1, \ldots, r$, it follows that $P_j = h_j(\varphi_1, \ldots, \varphi_n)$, where h_j is a partially defined C^k function on $\mathbb{R}^n$. Let $x_0 := \Phi(\emptyset_{t_0})$ (for any $t_0 \in \mathbb{T}$). Observe that $x_0 \in X$ and that all the functions $f_{\omega,i}$ and h_j are defined on open subsets of $\mathbb{R}^n$ containing X – the image of Φ. Consider a constant control $u = [\omega, t_0, t_1] \in U_P$. Let $u_t = [\omega, t_0, t]$ for $t \in [t_0, t_1]$. If $\gamma(t) = \Phi(u_t)$, then we have

$$\gamma(t_0) = \Phi(\emptyset_{t_0}) = x_0$$

and for any $s \in [t_0, t_1]$

$$\begin{aligned}
\gamma^\Delta(s) &= \frac{\Delta}{\Delta t} \,|_{t=s}\, \gamma(t) \\
&= \frac{\Delta}{\Delta t} \,|_{t=s}\, (\varphi_1([\omega, t_0, t]), \ldots, \varphi_n([\omega, t_0, t])) \\
&= \frac{\Delta}{\Delta t} \,|_{t=s+}\, (\varphi_1([\omega, t_0, s] \sqcup [\omega, s, t]), \ldots, \\
&\qquad \varphi_n([\omega, t_0, s] \sqcup [\omega, s, t])) \\
&= ((\Delta_\omega \varphi_1)(u_s), \ldots, (\Delta_\omega \varphi_n)(u_s)) \\
&= (f_{\omega,1}(\varphi_1, \ldots, \varphi_n), \ldots, f_{\omega,n}(\varphi_1, \ldots, \varphi_n))(u_s) \\
&= (f_{\omega,1}, \ldots, f_{\omega,n})(\gamma(s)).
\end{aligned}$$

So, the map γ is defined for all $t \in [t_0, t_1]$ and $\gamma(t) = \psi(t, x_0, u)$. Thus $u \in U_{\Sigma,x_0}$. Moreover

$$P_{\Sigma,x_0}(u) = h(\psi(t_1, x_0, u)) = h(\gamma(t_1)) \\ = h(\varphi_1(u), \dots, \varphi_n(u)) = P(u).$$

Similarly for piecewise constant controls from U_P.

"⇒" Any C^k real function φ defined on an open subset of $\mathbb{R}^n$ can be expressed as $\varphi = \varphi \circ (x^1, \dots, x^n)$, where $x^i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, n$, are the coordinate functions. Consider a realization of P given by a system Σ on $X \subseteq \mathbb{R}^n$ and an initial point $x_0 \in X$. Let $\varphi_i := x^i \circ \xi = \xi_i$, for $i = 1, \dots, n$, where ξ is given in Lemma 4.4. Observe that $\varphi_1, \dots, \varphi_n$ are defined on U_P. Lemma 4.4 implies that

$$(\Delta_\omega \varphi_i)(u) = [\Delta_\omega(x^i \circ \xi)](u) = [(\Gamma_{f_\omega}^{T_1^u} x^i) \circ \xi](u) \\ = f_{\omega,i}(\xi(u)) = f_{\omega,i}(\varphi_1, \dots, \varphi_n)(u)$$

so $\Delta_\omega(\varphi_i) \in C^k(\varphi_1, \dots, \varphi_n)$. Moreover

$$P_i(u) = (h_i \circ \xi)(u) = h_i(\varphi_1, \dots, \varphi_n)(u)$$

so $P_i \in C^k(\varphi_1, \dots, \varphi_n)$. ■

Remark 4.7: The sufficiency part of the proof of Theorem 4.6 gives a simple construction of the realization. If the domains of $f_{\omega,i}$ and h_j are the entire $\mathbb{R}^n$, then the constructed system is global. But the state space X, also constructed in the proof, may be much smaller. In general it is not a submanifold of $\mathbb{R}^n$, so one may be tempted to take rather $\mathbb{R}^n$ as the state space of the system. On the other hand, it is clear from the proof that the constructed system is controllable from the initial point, i.e. every $x \in X$ can be reached from x_0 by using a piecewise constant control from U_P. Thus X is better as a state space from this point of view.

Example 4.8: Let $\Omega = \mathbb{R}$, $U_P = U$ and $(Su)(t) = u(t)(t - T_0^u)$ for $u \in U_P$ and $t \in [T_0^u, T_1^u]$. Let the response map be given by $P(u) = e_{Su}(T_1^u, T_0^u)$ for $u \in U_P$. Then for $\omega \in \mathbb{R}$

$$(\Delta_\omega P)(u) = \frac{\Delta}{\Delta t}_{|t=T_1^u+} e_{S(u \sqcup (\omega, T_1^u, t))}(t, T_0^u) \\ = \frac{\Delta}{\Delta t}_{|t=T_1^u+} e_{Su}(T_1^u, T_0^u) e_{S(\omega, T_1^u, t)}(t, T_1^u) \\ = \omega P(u)(T_1^u - T_0^u).$$

Let $T : U_P \to \mathbb{R}$, $T(u) := T_1^u - T_0^u$. It is easy to see that $\Delta_\omega T = 1$ for every $\omega \in \Omega$. Let $\varphi_1 = P$, $\varphi_2 = T$ and $\Phi = (\varphi_1, \varphi_2)$. Then $\Delta_\omega \varphi_1 = \omega \varphi_1 \varphi_2$, $\Delta_\omega \varphi_2 = 1$ and $P = \varphi_1$. Moreover $\Phi(\emptyset_{t_0}) = (1, 0)$. Thus the realization of P is given by

$$x_1^\Delta = x_1 x_2 u \\ x_2^\Delta = 1 \\ y = x_1$$

with the initial condition $x(0) = (1, 0)$. The system is defined on the whole $\mathbb{R}^2$, but one can take $X = \Phi(U_P)$ as a state space. This, however, depends on the time scale $\mathbb{T}$.

Remark 4.9: As the above example shows, to find functions $\varphi_1, \dots, \varphi_n$ needed for the construction of the realization, we first use the components of P and then compute their derivatives $\Delta_\omega P_i$ looking for other functions.

Remark 4.10: A response map may have many different realizations. One usually looks for minimal ones, which are controllable and observable in some sense. This was done for linear systems on time scales in [12]. The nonlinear case is more complicated as controllability and observability for systems on arbitrary time scales have not been studied yet. Moreover there are many different controllability and observability concepts for nonlinear systems (even for $\mathbb{T} = \mathbb{R}$), so minimality may have several meanings. If we allow for the state space of the system to be an arbitrary subset of $\mathbb{R}^n$, then constructing controllable realization is always possible (such a realization is constructed in the proof of Theorem 4.6). To obtain an observable system from an unobservable one, we need to identify the states that are indistinguishable. This may lead to a system with a state space that is no longer a subset of $\mathbb{R}^n$. In [16] universal spaces were proposed to deal with this problem for analytic continuous-time systems. However it is not clear how to extend this to systems on arbitrary time scales.

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