Noether's Theorem on Time Scales ⋆

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Abstract

We show that for any variational symmetry of the problem of the calculus of variations on time scales there exists a conserved quantity along the respective Euler-Lagrange extremals.

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1 Introduction

The calculus on time scales is a relatively new theory that unifies and generalizes difference and differential equations. The theory was initiated by Stefan

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Hilger and is being applied to many different fields in which dynamic processes can be described with discrete or continuous models [1,8,14]. The calculus of variations on time scales was initiated in 2004 with the papers [6,15].

The calculus of variations and control theory are disciplines in which there appears to be many opportunities for application of time scales [4,5,10]. Here we make use of the Euler-Lagrange equations on time scales [6,9,15] to generalize one of the most beautiful results of the calculus of variations – the celebrated Noether's theorem [13,18,19]. Our Noether-type theorem (Theorem 4) unifies and extends the previous formulations of Noether's principle in the discrete-time and continuous domains (cf. [21,22] and references therein). Moreover, it gives answer (Corollary 2) to an open question formulated in [17, p. 216] (see also [21, Remark 12]): *how to obtain 'energy' integrals for discrete-time problems, as done in the continuous calculus of variations?*

2 Preliminaries on time scales

We give here basic definitions and facts concerning the calculus on time scales. More information can be found e.g. in [8].

A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. It is a model of time. Besides standard cases of $\mathbb{R}$ (continuous time) and $\mathbb{Z}$ (discrete time), many different models are used. For each time scale $\mathbb{T}$ the following operators are used:

- the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ for $t < \sup \mathbb{T}$ and $\sigma(\sup \mathbb{T}) = \sup \mathbb{T}$ if $\sup \mathbb{T} < +\infty$;
- the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$, $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ for $t > \inf \mathbb{T}$ and $\rho(\inf \mathbb{T}) = \inf \mathbb{T}$ if $\inf \mathbb{T} > -\infty$;
- the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$, $\mu(t) := \sigma(t) - t$.

Example 1 *If $\mathbb{T} = \mathbb{R}$, then for any $t \in \mathbb{R}$, $\sigma(t) = t = \rho(t)$ and $\mu(t) \equiv 0$. If $\mathbb{T} = \mathbb{Z}$, then for every $t \in \mathbb{Z}$, $\sigma(t) = t + 1$, $\rho(t) = t - 1$ and $\mu(t) \equiv 1$.*

A point $t \in \mathbb{T}$ is called: (i) *right-scattered* if $\sigma(t) > t$, (ii) *right-dense* if $\sigma(t) = t$, (iii) *left-scattered* if $\rho(t) < t$, (iv) *left-dense* if $\rho(t) = t$, (v) *isolated* if it is both left-scattered and right-scattered, (vi) *dense* if it is both left-dense and right-dense. If $\sup \mathbb{T}$ is finite and left-scattered, we set $\mathbb{T}^\kappa := \mathbb{T} \setminus \{\sup \mathbb{T}\}$. Otherwise, $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 1 *Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. The delta derivative of f at t is the real number $f^\Delta(t)$ with the property that given any ε there is a neighborhood*

$U = (t - \delta, t + \delta) \cap \mathbb{T}$ *of* t *such that*

$$|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in U$. *We say that* f *is delta differentiable on* $\mathbb{T}$ *provided* $f^{\Delta}(t)$ *exists for all* $t \in \mathbb{T}^{\kappa}$.

We shall often denote $f^{\Delta}(t)$ by $\frac{\Delta}{\Delta t} f(t)$ if f is a composition of other functions. The delta-derivative of a function $f : \mathbb{T} \to \mathbb{R}^n$ is a (column) vector whose components are delta-derivatives of the components of f. For $f : \mathbb{T} \to X$, where X is an arbitrary set, we define $f^{\sigma} := f \circ \sigma$.

Remark 1 *If* $\mathbb{T} = \mathbb{R}$, *then* $f : \mathbb{R} \to \mathbb{R}$ *is delta differentiable at* $t \in \mathbb{R}$ *if and only if* f *is differentiable in the ordinary sense at* t. *Then,* $f^{\Delta}(t) = f'(t)$. *If* $\mathbb{T} = \mathbb{Z}$, *then* $f : \mathbb{Z} \to \mathbb{R}$ *is always delta differentiable at every* $t \in \mathbb{Z}$ *with* $f^{\Delta}(t) = f(t+1) - f(t)$.

Definition 2 *A function* $f : \mathbb{T} \to \mathbb{R}$ *is called* rd-continuous *if it is continuous at the right-dense points in* $\mathbb{T}$ *and its left-sided limits exist at all left-dense points in* $\mathbb{T}$. *A function* $f : \mathbb{T} \to \mathbb{R}^n$ *is* rd-continuous *if all its components are rd-continuous.*

The set of all rd-continuous functions is denoted by $\mathcal{C}_{rd}$. Similarly, $\mathcal{C}^1_{rd}$ will denote the set of functions from $\mathcal{C}_{rd}$ whose delta derivative belongs to $\mathcal{C}_{rd}$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an *antiderivative of* $f : \mathbb{T} \to \mathbb{R}$ if it satisfies $F^{\Delta}(t) = f(t)$, for all $t \in \mathbb{T}^{\kappa}$. Then, the *indefinite integral* of f is defined by $\int f(t)\Delta t = F(t) + C$, where C is an arbitrary constant. The *definite integral* of f is defined by $\int_r^s f(t)\Delta t = F(s) - F(r)$, for all $s, t \in \mathbb{T}$. It is known that every rd-continuous function has an antiderivative.

Example 2 *If* $\mathbb{T} = \mathbb{R}$, *then*

$$\int_a^b f(t)\Delta t = \int_a^b f(t)dt,$$

where the integral on the right hand side is the usual Riemann integral. If $\mathbb{T} = h\mathbb{Z}$, *where* $h > 0$, *then*

$$\int_a^b f(t)\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} h \cdot f(kh),$$

for $a < b$.

We shall need the following properties of delta derivatives and integrals:

$$(fg)^\Delta = f^\Delta g^\sigma + f g^\Delta \,, \tag{1}$$

$$f^\sigma = f + \mu f^\Delta \,, \tag{2}$$

$$\int_a^b f(\alpha(t))\alpha^\Delta(t)\Delta t = \int_{\alpha(a)}^{\alpha(b)} f(\bar{t})\bar{\Delta}\bar{t} \,, \tag{3}$$

where $\alpha : [a,b] \cap \mathbb{T} \rightarrow \mathbb{R}$ is an increasing C^1_{rd} function and its image is a new time scale (so a new symbol $\bar{\Delta}\bar{t}$ is used in the second integral).

3 Main results

There exist several different ways to prove the classical theorem [18] of Emmy Noether. Three different proofs of the classical Noether's theorem are reviewed in [11, Chap. 1], a fourth one can be found in [13]. In this section we follow one of those proofs, which is based on a technique of time-reparameterization [16]. While this approach is not so popular for proving the classical Noether's theorem and many of its extensions (see e.g. [13,22,23]), it has already shown to be very effective in two generalizations of the classical result: it has been used in [20] in order to get a more general Noether's theorem in the optimal control setting; it has been used in [12] to deal with problems of the calculus of variations with fractional derivatives in the Riemann-Liouville sense. Here we use this technique to prove a Noether-type theorem for problems of the calculus of variations on time scales.

We consider the fundamental problem of the calculus of variations on time scales as defined by Bohner [6] (see also [2,3,15]):

$$I[q(\cdot)] = \int_a^b L\left(t, q^\sigma(t), q^\Delta(t)\right) \Delta t \longrightarrow \min, \tag{4}$$

under given boundary conditions $q(a) = q_a$, $q(b) = q_b$, where $q^\sigma(t) = (q \circ \sigma)(t)$, $q^\Delta(t)$ is the *delta derivative*, $t \in \mathbb{T}$, and the Lagrangian $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a C^1 function with respect to its arguments. By $\partial_i L$ we will denote the partial derivative of L with respect to the i-th variable, $i = 1, 2, 3$. Admissible functions $q(\cdot)$ are assumed to be C^1_{rd}.

Theorem 1 (See [6]) *If $q(\cdot)$ is a minimizer of problem* (4)*, then $q(\cdot)$ satisfies the following* Euler-Lagrange equation*:*

$$\frac{\Delta}{\Delta t}\partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) = \partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right) \,. \tag{5}$$

Definition 3 (invariance without transforming the time) *Let U be a set of C^1_{rd} functions $q : [a,b] \to \mathbb{R}^n$. The functional I is said to be invariant on U*

under a one-parameter family of state transformations

$$\bar{q} = q + \varepsilon\xi(t,q) + o(\varepsilon) \tag{6}$$

if, and only if,

$$\int_{t_a}^{t_b} L\left(t, q^\sigma(t), q^\Delta(t)\right)\Delta t = \int_{t_a}^{t_b} L\left(t, \bar{q}^\sigma(t), \bar{q}^\Delta(t)\right)\Delta t \tag{7}$$

for any subinterval $[t_a, t_b] \subseteq [a,b]$ *with* $t_a, t_b \in \mathbb{T}$, *for any* ε *and for any* $q \in U$, *where* $\bar{q}(t) = q(t) + \varepsilon\xi(t, q(t)) + o(\varepsilon)$.

Definition 4 (conservation law) *Quantity* $C(t, q, q^\sigma, q^\Delta)$ *is said to be a* conservation law *for functional* I *on* U *if, and only if,* $\frac{\Delta}{\Delta t}C(t, q(t), q^\sigma(t), q^\Delta(t)) = 0$ *along all* $q \in U$ *that satisfy the Euler-Lagrange equation* (5).

Theorem 2 (necessary condition of invariance) *If functional* I *is invariant on* U *under transformations* (6), *then*

$$\partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\sigma(t, q(t)) + \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\Delta(t, q(t)) = 0 \tag{8}$$

for all $t \in [a,b]$ *and all* $q \in U$, *where* $\xi^\sigma(t, q(t)) = \xi(\sigma(t)), q(\sigma(t)))$ *and* $\xi^\Delta(t, q(t)) = \frac{\Delta}{\Delta t}\xi(t, q(t))$.

Proof: Having in mind that condition (7) is valid for any subinterval $[t_a, t_b] \subseteq [a, b]$, we can rid off the integral signs in (7): equation (7) is equivalent to

$$L\left(t, q^\sigma(t), q^\Delta(t)\right) = L(t, q^\sigma(t) + \varepsilon\xi^\sigma(t, q(t)) + o(\varepsilon), q^\Delta(t) + \varepsilon\xi^\Delta(t, q(t)))\,. \tag{9}$$

Differentiating both sides of equation (9) with respect to ε, then setting $\varepsilon = 0$, we obtain equality (8). $\Box$

Theorem 3 (Noether's theorem without transforming time) *If functional* I *is invariant on* U *under the one-parameter family of transformations* (6), *then*

$$C(t, q, q^\sigma, q^\Delta) = \partial_3 L\left(t, q^\sigma, q^\Delta\right) \cdot \xi(t, q) \tag{10}$$

is a conservation law.

Proof: Using the Euler-Lagrange equation (5) and the necessary condition of invariance (8), we obtain:

$$\begin{aligned}
&\frac{\Delta}{\Delta t}\left(\partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi(t, q(t))\right) \\
&\quad = \frac{\Delta}{\Delta t}\partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\sigma(t, q(t)) + \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\Delta(t, q(t)) \\
&\quad = \partial_2 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\sigma(t, q(t)) + \partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right) \cdot \xi^\Delta(t, q(t)) \\
&\quad = 0\,.\Box
\end{aligned}$$

Remark 2 *In classical mechanics, for* $\mathbb{T} = \mathbb{R}$, $\partial_3 L\left(t, q^\sigma(t), q^\Delta(t)\right)$ *is interpreted as the generalized momentum.*

Let us consider now the one-parameter family of infinitesimal transformations

$$\begin{cases} \bar{t} = T_\varepsilon(t,q) = t + \varepsilon\tau(t,q) + o(\varepsilon)\,, \\ \bar{q} = Q_\varepsilon(t,q) = q + \varepsilon\xi(t,q) + o(\varepsilon)\,. \end{cases} \tag{11}$$

For a fixed ε we will drop this index in T_ε and Q_ε and write T and Q instead. Let as before U be a set of C^1_{rd} functions $q : [a,b] \to \mathbb{R}^n$. We assume that for every $q \in U$ and every ε, the map $[a,b] \ni t \mapsto \alpha(t) := T_\varepsilon(t, q(t)) \in \mathbb{R}$ is an increasing C^1_{rd} function and its image is again a time scale with the forward shift operator $\bar{\sigma}$ and the delta derivative $\bar{\Delta}$. Observe that the following holds:

$$\bar{\sigma} \circ \alpha = \alpha \circ \sigma. \tag{12}$$

Let $\beta = \alpha^{-1}$. We set $\bar{q}(\bar{t}) := Q\left(\beta(\bar{t}), q(\beta(\bar{t}))\right)$.

Definition 5 (invariance of I) *Functional I is said to be invariant on U under the family of transformations* (11) *if, and only if, for any subinterval* $[t_a, t_b] \subseteq [a,b]$*, any ε and any $q \in U$*

$$\int_{t_a}^{t_b} L\left(t, q^\sigma(t), q^\Delta(t)\right)\Delta t = \int_{T(t_a,q(t_a))}^{T(t_b,q(t_b))} L\left(\bar{t}, \bar{q}^{\bar{\sigma}}(\bar{t}), \bar{q}^{\bar{\Delta}}(\bar{t})\right)\bar{\Delta}\bar{t}.$$

Remark 3 *Observe that in Definition 5 we change time. Thus, we consider the functional I on many different time scales, depending on ε and $q(\cdot)$. This is the reason for assuming that the Lagrangian L is defined for all $t \in \mathbb{R}$ and not just t from the initial time scale $\mathbb{T}$.*

Theorem 4 (Noether's theorem) *If functional I is invariant on U, in the sense of Definition 5, then*

$$\begin{aligned} C(t,q,q^\sigma,q^\Delta) &= \partial_3 L\left(t,q^\sigma,q^\Delta\right)\cdot\xi(t,q) \\ &\quad + \left[L(t,q^\sigma,q^\Delta) - \partial_3 L\left(t,q^\sigma,q^\Delta\right)\cdot q^\Delta - \partial_1 L\left(t,q^\sigma,q^\Delta\right)\cdot\mu(t)\right]\cdot\tau(t,q) \end{aligned} \tag{13}$$

is a conservation law.

Proof: We will show that invariance of I under (11) (in the sense of Definition 5) is equivalent to invariance of another functional $\tilde{I}$ in the sense of Definition 3.

Let $\tilde{L}(t; s, q; r, v) := L(s - \mu(t)r, q, \frac{v}{r}) \cdot r$ for $q, v \in \mathbb{R}^n$, $t \in [a,b]$ and $s, r \in \mathbb{R}$, $r \neq 0$. $\tilde{L}$ is a new Lagrangian with the state variable $(s,q) \in \mathbb{R}^{n+1}$. Observe that for $s(t) = t$ and any $q : [a,b] \to \mathbb{R}^n$

$$L(t, q^\sigma(t), q^\Delta(t)) = \tilde{L}(t; s^\sigma(t), q^\sigma(t); s^\Delta(t), q^\Delta(t))$$

so for the functional

$$\tilde{I}[s(\cdot), q(\cdot)] := \int_a^b \tilde{L}(t; s^\sigma(t), q^\sigma(t); s^\Delta(t), q^\Delta(t))\Delta t$$

we get $I[q(\cdot)] = \tilde{I}[s(\cdot), q(\cdot)]$ whenever $s(t) = t$.

Consider the group of transformations $(T_\varepsilon, Q_\varepsilon)$ given by (11) and let $q \in U$. From the invariance of I, for $s(t) = t$, we get

$$\begin{aligned}
\tilde{I}[s(\cdot), q(\cdot)] &= I\left[q(\cdot)\right] = \int_a^b L\left(t, q^\sigma(t), q^\Delta(t)\right)\Delta t \\
&= \int_{\alpha(a)}^{\alpha(b)} L\left(\bar{t}, (\bar{q}\circ\bar{\sigma})(\bar{t}), \bar{q}^{\bar{\Delta}}(\bar{t})\right)\bar{\Delta}\bar{t} \\
&= \int_a^b L\left(\alpha(t), (\bar{q}\circ\bar{\sigma}\circ\alpha)(t), \bar{q}^{\bar{\Delta}}(\alpha(t))\right)\alpha^\Delta(t)\Delta t \\
&= \int_a^b L\left(\alpha^\sigma(t) - \mu(t)\alpha^\Delta(t), (\bar{q}\circ\alpha)^\sigma(t), \frac{(\bar{q}\circ\alpha)^\Delta(t)}{\alpha^\Delta(t)}\right)\alpha^\Delta(t)\Delta t \\
&= \int_a^b \tilde{L}\left(t; \alpha^\sigma(t), (\bar{q}\circ\alpha)^\sigma(t); \alpha^\Delta(t), (\bar{q}\circ\alpha)^\Delta(t)\right)\right)\Delta t \\
&= \tilde{I}[\alpha(\cdot), (\bar{q}\circ\alpha)(\cdot)].
\end{aligned}$$

Observe that for $s(t) = t$

$$(\alpha(t), (\bar{q}\circ\alpha)(t)) = (T_\varepsilon(t, q(t)), Q_\varepsilon(t, q(t))) = (T_\varepsilon(s(t), q(t)), Q_\varepsilon(s(t), q(t))).$$

This means that $\tilde{I}$ is invariant on $\tilde{U} = \{(s, q) \mid s(t) = t,\ q \in U\}$ under the group of state transformations

$$(\bar{s}, \bar{q}) = (T_\varepsilon(s, q), Q_\varepsilon(s, q))$$

in the sense of Definition 3. Applying Theorem 3, we obtain that for $s(t) = t$

$$\begin{aligned}
C\left(t, s, q, s^\sigma, q^\sigma, s^\Delta, q^\Delta\right) &= \partial_5\tilde{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta)\cdot\xi(s, q) \\
&\quad + \partial_4\tilde{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta)\cdot\tau(s, q) \qquad (14)
\end{aligned}$$

is a conservation law. Since

$$\partial_5\bar{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) = \partial_3 L\left(s^\sigma - \mu(t)s^\Delta, q^\sigma, \frac{q^\Delta}{s^\Delta}\right)$$

and

$$\partial_4 \bar{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) = -\partial_1 L\left(s^\sigma - \mu(t)s^\Delta, q^\sigma, \frac{q^\Delta}{s^\Delta}\right) \cdot \mu(t) \cdot s^\Delta$$
$$- \partial_3 L\left(s^\sigma - \mu(t)s^\Delta, q^\sigma, \frac{q^\Delta}{s^\Delta}\right) \cdot \frac{q^\Delta}{s^\Delta} + L\left(s^\sigma - \mu(t)s^\Delta, q^\sigma, \frac{q^\Delta}{s^\Delta}\right),$$

for $s(t) = t$ we get

$$\partial_5 \bar{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta) = \partial_3 L\left(t, q^\sigma, q^\Delta\right) \tag{15}$$

and

$$\partial_4 \bar{L}(t; s^\sigma, q^\sigma; s^\Delta, q^\Delta)$$
$$= L(t, q^\sigma, q^\Delta) - \partial_3 L\left(t, q^\sigma, q^\Delta\right) \cdot q^\Delta - \partial_1 L\left(t, q^\sigma, q^\Delta\right) \cdot \mu(t). \tag{16}$$

Substituting (15) and (16) into (14) we arrive to the intended conclusion (13). □

For $\mathbb{T} = \mathbb{R}$ the formula (13) simplifies due to the fact that $\mu \equiv 0$, and we obtain the classical Noether's theorem:

Corollary 1 *Let $\mathbb{T} = \mathbb{R}$. If functional I is invariant on U, in the sense of Definition 5, then*

$$C(t, q, q') = \partial_3 L\left(t, q, q'\right) \cdot \xi(t, q) + \left[L(t, q, q') - \partial_3 L\left(t, q, q'\right) \cdot q'\right] \cdot \tau(t, q)$$

is a conservation law.

Remark 4 *In classical mechanics, the term $L(t, q, q') - \partial_3 L\left(t, q, q'\right) \cdot q'$ is interpreted as the energy.*

For the discrete-time case ($\mathbb{T} = \mathbb{Z}$), we obtain a new version of Noether's theorem which generalizes the result in [21]:

Corollary 2 *Let $\mathbb{T} = \mathbb{Z}$. If functional I is invariant on U, in the sense of Definition 5, then*

$$C(t, q, q^+, \Delta q) = \partial_3 L\left(t, q^+, \Delta q\right) \cdot \xi(t, q)$$
$$+ \left[L(t, q^+, \Delta q) - \partial_3 L\left(t, q^+, \Delta q\right) \cdot \Delta q - \partial_1 L\left(t, q^+, \Delta q\right)\right] \cdot \tau(t, q)$$

is a conservation law, where $q^+(t) = q(t+1)$ and $\Delta q = q^+ - q$.

We finish with an example of a conservation law on a discrete but nonhomogeneous time scale (graininess is not constant).

Example 3 *Let* $\mathbb{T} = \{2^n : n \in \mathbb{N} \cup \{0\}\}$ *and*

$$L(t, q^\sigma, q^\Delta) = \frac{(q^\sigma)^2}{t} + t(q^\Delta)^2$$

for $q \in \mathbb{R}$. *It can be shown that the functional* I *is invariant under the family of transformations:*

$$\bar{t} = te^\varepsilon = t + t\varepsilon + o(\varepsilon), \;\; \bar{q} = q.$$

Then, Noether's theorem generates the following conservation law:

$$C(t, q^\sigma, q^\Delta) = 2\left[\frac{(q^\sigma)^2}{t} - t(q^\Delta)^2\right] \cdot t.$$

References

[1] R. Agarwal, M. Bohner, D. O'Regan, A. Peterson. *Dynamic equations on time scales: a survey*, J. Comput. Appl. Math. 141 (2002) 1–26.

[2] C.D. Ahlbrandt, M. Bohner, J. Ridenhour. *Hamiltonian systems on time scales*, J. Math. Anal. Appl. 250 (2000) 561–578.

[3] F.M. Atici, D.C. Biles, A. Lebedinsky. *An application of time scales to economics*, Math. Comput. Modelling 43 (2006) 718–726.

[4] Z. Bartosiewicz, Ü. Kotta, E. Pawłuszewicz. *Equivalence of linear control systems on time scales*, Proc. Estonian Acad. Sci. Phys. Math. 55 (2006) 43–52.

[5] Z. Bartosiewicz, E. Pawłuszewicz. *Dynamic feedback equivalence of nonlinear systems on time scales*, Proceedings of 16th IFAC World Congress, 4-8 July 2005, Prague.

[6] M. Bohner. *Calculus of variations on time scales*, Dynam. Systems Appl. 13 (2004) 339–349.

[7] M. Bohner, G.Sh. Guseinov. *Double integral calculus of variations on time scales*, Computers and Mathematics with Applications, 54 (2007) 45–57.

[8] M. Bohner, A. Peterson. *Dynamic equations on time scales – an introduction with applications*, Birkhäuser Boston, Inc., Boston, MA, 2001.

[9] R.A.C. Ferreira, D.F.M. Torres. *Higher-order calculus of variations on time scales*, Proceedings of the Workshop on Mathematical Control Theory and Finance, Lisbon, 10-14 April 2007, 150–158.

[10] R.A.C. Ferreira, D.F.M. Torres. *Remarks on the calculus of variations on time scales*, Int. J. Ecol. Econ. Stat. 9 (2007) 65–73.

[11] G.S.F. Frederico. *Extensions of Noether's theorem in the calculus of variations and optimal control* (in Portuguese), Ph.D. thesis, University of Cape Verde, 2007.

[12] G.S.F. Frederico, D.F.M. Torres. *A formulation of Noether's theorem for fractional problems of the calculus of variations*, J. Math. Anal. Appl. 334 (2007) 834–846.

[13] I.M. Gelfand, S.V. Fomin. *Calculus of variations*, Dover, New York, 2000.

[14] S. Hilger. *Differential and difference calculus—unified!*, Proceedings of the Second World Congress of Nonlinear Analysts, Part 5 (Athens, 1996). Nonlinear Anal. 30 (1997) 2683–2694.

[15] R. Hilscher, V. Zeidan. *Calculus of variations on time scales: weak local piecewise C^1_{rd} solutions with variable endpoints*, J. Math. Anal. Appl. 289 (2004) 143–166.

[16] J. Jost, X. Li-Jost. *Calculus of variations*. Cambridge Studies in Advanced Mathematics, 64. Cambridge University Press, Cambridge, 1998.

[17] J.D. Logan. *First integrals in the discrete variational calculus*, Aequationes Math. 9 (1973), 210–220.

[18] E. Noether. *Invariant variation problems*, Transport Theory Statist. Phys. 1(3), 186–207 [Translated from the German (Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1918, 235–257)], 1971.

[19] W. Sarlet, F. Cantrijn. *Generalizations of Noether's theorem in classical mechanics*, SIAM Rev. 23 (1981) 467–494.

[20] D.F.M. Torres. *On the Noether theorem for optimal control*, European Journal of Control **8** (2002) 56–63.

[21] D.F.M. Torres. *Integrals of motion for discrete-time optimal control problems*, Control Applications of Optimisation 2003 (Eds: R. Bars, E. Gyurkovics), IFAC Workshop Series 2003, 33–38.

[22] D.F.M. Torres. *Proper extensions of Noether's symmetry theorem for nonsmooth extremals of the calculus of variations*, Commun. Pure Appl. Anal. 3 (2004) 491–500.

[23] B. van Brunt. *The calculus of variations*, Universitext, Springer-Verlag, New York, 2004.