Overview of viability results *

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Abstract

Let K be a multivalued map from the interval $[0,1]$ to $\mathbb{R}^n$. On its graph, denoted by $Gr(K)$, another multivalued map F is defined. Its values are closed subsets of $\mathbb{R}^n$. The map F, called an *orientor field*, gives rise to a differential inclusion $\dot{x}(t) \in F(t, x(t))$. The inclusion is called *viable*, if for every $(t_0, x_0) \in Gr(K)$ there is a global absolutely continuous forward trajectory $x : [t_0, 1] \to \mathbb{R}^n$ of this inclusion, satisfying the initial condition $x(t_0) = x_0$. As the differential inclusion may come from a control system $\dot{x}(t) = f(t, x(t), u(t))$, the viability of the differential inclusion may be interpreted as a kind of controllability of the control system with time dependent constraints. The first result on viability, by N. Nagumo, was formulated for a constant multifunction K and a single-valued time-independent F. In its full complexity, viability was studied by J.-P. Aubin, D. Bothe, K. Deimling, H. Frankowska, S. Hu, N. Papageorgiu, S. Plaskacz, T. Rzeżuchowski, and many others. The obtained results give conditions on K and F that guarantee the viability of the inclusion. Besides various measurability and continuity assumptions, the essential one states that the intersection of $F(t, x)$ with some generalized derivative of K at (t, x) is nonempty for almost all $t \in [0,1]$ and all $x \in K(t)$. This requirement is a direct extension of the Nagumo condition. We recall all these conditions and present our own viability criteria. As a generalized derivative we choose Generalized Differential Quotient (GDQ), introduced recently by H. Sussmann. Since GDQ of a multivalued function is not unique, we use one, denoted by SGDQ, that seems to fit best. SGDQ is the closure of the union of all minimal GDQs. Finally, we compare conditions introduced by different authors, setting a road map of the viability problem.

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1 Introduction

Jednym z zastosowa teorii uoglnionych pochodnych jest badanie istnienia rozwiza inkluzji rniczkowych, w ktrych multiwartociowe pola rniczkowe (zwane te orientorowymi) okrelone s na domknitych zbiorach. Drono inkluzji rniczkowych to problem istnienia globalnych rozwiza inkluzji rniczkowych, ktrych dynamiki s obcite do domknitego podzbioru przestrzeni stanw. Pierwszym rezultatem w tej dziedzinie jest twierdzenie Nagumo z 1942 roku. Sformuowa on konieczny i wystarczajcy warunek na to, aby wszystkie trajektorie pola wektorowego startujce z punktw nalecych do domknitego zbioru K pozostaway w tym zbiorze.

Jeli teraz zastpimy rwnanie rniczkowe przez inkluzj rniczkow, to wwczas, co jest dosy oczywiste, stracimy jednoznaczno rozwiza. Rozwaa mona wtedy dwa problemy: przy jakich warunkach wszystkie trajektorie startujce z dowolnego punktu ze zbioru K pozostaj w tym zbiorze (tzw. niezmienniczo) albo kiedy dla kadego punktu zbioru K przynajmniej jedna trajektoria startujca z tego punktu pozostanie w K (tzw. drono). Innym rozwiniciem problemu jest rozwaanie czaso-zalenych inkluzji rniczkowych okrelonych na domknitych zbiorach rwnie zmieniajcych si w czasie.

Niech $T = [0, b]$ i dla kadego $t \in T$, niech $K(t)$ bdzie podzbiorem $\mathbb{R}^n$. Zatem K jest multifunkcj, ktrej wykres GrK skada sie z punktw $(t, y) \in \mathbb{R} \times \mathbb{R}^n$, takich e $t \in T$ i $y \in K(t)$. Czaso-zalene pole orientorowe ograniczone przez K jest multifunkcj F zdefiniowan na GrK, ktrej wartoci s podzbiorami $\mathbb{R}^n$. Wtedy wielowartociowy problem Cauchy'ego jest zdefiniowany nastepujco:

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{a.e. on T} \\ y(t_0) = y_0. \end{cases} \tag{1}$$

Przedmiotem naszego zainteresowania s warunki, ktre zapewni istnienie rozwizania $y : T \to \mathbb{R}^n$ problemu Cauchy'ego (1). Dla F zdefiniowanego tylko na GrK, y musi spenia warunek $y(t) \in K(t)$ dla wszystkich $t \in T$.

Powyszy problem, chocia dosy nowy, rozwaany by przez wiele osb. Wydaje si, e najistotniejszymi pracami dotyczcymi tej tematyki s artykuy D. Bothego [3], H. Frankowskiej, S. Plaskacza, T. Rzeżuchowskiego [5] oraz rozdziay ksiek autorstwa Sh. Hu i S.N. Papageorgiou [10], a take P. Aubin'a i A. Celliny [1].

W naszym podejciu do problemu dronoci inkluzji rniczkowych zamiast, standardowej ju, pochodnej kontyngensowej pojawia si oprcz innych zaoe, GDQ rzniczkowalno multifunkcji K. To nowe, i jak si wydaje, naturalne podejcie do problemu daje ciekawe wyniki przedstawione w Twierdzeniu-? i ?.

def lewej usc def lewo absolutnie ciaglych

2 Basic notations and definitions

By a set-valued map (multifunction) we mean a triple $F = (A, B, G)$ such that A and B are sets and G is a subset of $A \times B$. The sets A, B, G are, respectively, the *source*, *target* and *graph* of F, and we write $A = So(F)$, $B = Ta(F)$, $G = Gr(F) := \{(x, y) : y \in F(x)\}$. For $x \in So(F)$ we write $F(x) = \{y :$

$(x, y) \in Gr(F)\}$ (it can happen that $F(x) = \emptyset$ for $x \in So(F)$). The sets $Do(F) = \{x \in So(F) : F(x) \neq \emptyset\}$, $Im(F) = \bigcup_{x \in So(F)} F(x)$, are, respectively, the *domain* and *image* of F. If $F = (A, B, G)$ is a set-valued map, we say that F is a set-valued map from A to B with graph G, and write $F : A \twoheadrightarrow B$. We use $SVM(A, B)$ to denote the set of all set-valued maps from A to B. We reserve capital letters for set-valued maps and small ones for ordinary (single-valued and everywhere defined) maps.

If X is a metric space supplied with a distance d, $K \subseteq X$, then we denote the *distance from* x *to* K by $dist(x, K) := \inf_{y \in K} d(x, y)$, where we set $dist(x, \emptyset) := +\infty$. The *ball of radius* $\epsilon > 0$ *around* K *in* X is denoted by $B(K, \epsilon) := K^{\epsilon} := \{x \in X : dist(x, K) < \epsilon\}$. *Unit ball* is denoted by B. The balls $B(K, \epsilon)$ are neighborhoods of K. When K is compact, each neighborhood of K contains such a ball around K.

Let X and Y be metric spaces. We say that a set-valued map $F : X \twoheadrightarrow Y$ is *upper semicontinuous* (abbr. u.s.c.) at $\bar{x} \in Do(F)$ if and only if for any neighborhood U of $F(x)$ there exists $\delta > 0$ such that for every $x \in B(\bar{x}, \delta)$, $F(x) \subset U$. A set $C \subseteq \mathbb{R}^n$ is called a *cone* if $rx \in C$ for all $x \in C$ and $r \geq 0$. We say that F_n *graph converges* to F, and write $F_n \xrightarrow{gr} F$, if

$$\lim_{n \to \infty} \triangle(Gr(F_n), Gr(F)) = 0$$

where

$$\triangle(A, B) = \sup\{dist(q, B) : q \in A\}.$$

For $F \in SVM(\mathbb{R}^n, \mathbb{R}^m)$ we define $||F(x)|| := \sup\{||y|| : y \in F(x)\}$ if $F(x) \neq \emptyset$ and set $||\emptyset|| = -\infty$.
Let $\mathcal{T}$ be a metric space and $\{A_\tau\}_{\tau \in \mathcal{T}}$ be a family of subsets of a metric space X. The upper limit *Limsup* and the lower limit *Liminf* of A_τ at τ_0 are closed sets defined by

$$\text{Limsup}_{\tau \to \tau_0} A_\tau = \left\{ v \in X | \liminf_{\tau \to \tau_0} dist(v, A_\tau) = 0 \right\}$$

$$\text{Liminf}_{\tau \to \tau_0} A_\tau = \left\{ v \in X | \limsup_{\tau \to \tau_0} dist(v, A_\tau) = 0 \right\}.$$

A subset $A \subset X$ is said to be the limit of A_τ if

$$A = \text{Limsup}_{\tau \to \tau_0} = \text{Liminf}_{\tau \to \tau_0} =: \text{Lim}_{\tau \to \tau_0} A_\tau.$$

Let us consider a multifunction $K : T \twoheadrightarrow \mathbb{R}^n$, $Do(K) = T = [0, a] \subseteq \mathbb{R}$. We say that K is *left usc* if for every $t_0 \in (0, a]$ and $\varepsilon > 0$ there exists a $\delta = \delta(t_0, \varepsilon) > 0$ such that

$$K(t) \subset K(t_0) + B(0, \varepsilon) \quad \text{for all } t \in (t_0 - \delta, t_0] \cap T.$$

Let K be closed-valued. We say that K is left absolutely continuous on $[0, a]$ if the following property holds:

$$\forall\, \varepsilon > 0,\ \forall \text{ compact } P \subset \mathbb{R}^n,\ \exists \delta > 0,\ \forall N_0 \subset \mathbb{N}$$
$$\forall \{t_i, \tau_i : t_i < \tau_i, i \in N_0\} \text{ with } (t_i, \tau_i) \cap (t_j, \tau_j) = \emptyset \text{ for } i \neq j,$$
$$\Sigma(\tau_i - t_i) \le \delta \Rightarrow \Sigma \triangle (K(t_i) \cap P, K(\tau_i)) \le \varepsilon.$$

Let $K : T \twoheadrightarrow \mathbb{R}^n$, where $Do(K) = T = [0, a] \subseteq \mathbb{R}$, be a constraint multifunction and $F : GrK \twoheadrightarrow \mathbb{R}^n$, where $Do(F) = GrK$, be an orientor field (i.e. multivalued vector field). Consider the multivalued Cauchy problem as follows:

$$\begin{cases} \dot{y}(t) \in F(t, y(t)), & \text{a.e. on T} \\ y(t_0) = y_0. \end{cases} \tag{2}$$

By a *solution* $y(\cdot)$ to (2) we mean an absolutely continuous function $y : [t_0, a] \to \mathbb{R}^n$ that satisfies the inclusion almost everywhere and satisfies the initial condition. By a *viable solution* to (2) we mean a solution $y(\cdot)$ such that $y(t) \in K(t)$ for $t \in [t_0, a]$.

Definition 1. [2] A set-valued map $F : K \twoheadrightarrow Y$, where $K \subset X$ and $Do(F) = K$, has the *contingent derivative* $DF(x_0, y_0)$ at $x_0 \in K$ and $y_0 \in F(x_0)$ if $DF(x_0, y_0)$ is a set-valued map from X to Y whose graph is the *contingent cone* $T_{Gr(F)}(x_0, y_0)$ to the graph of F at (x_0, y_0). In other words,

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow (v_0, u_0) \in T_{Gr(F)}(x_0, y_0),$$

where the *contingent cone* (the "Bouligand cone") to C at x is defined by

$$T_C(x) = \left\{ w \in X : \liminf_{t \downarrow 0} \frac{dist(x + tw, C)}{t} = 0 \right\}.$$

Equivalently, we can write:

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow \liminf_{h \to 0^+ u \to u_0} \ dist\left(v_0, \frac{F(x_0 + hu) - y_0}{h} \right) = 0.$$

When F is a locally Lipschitz set-valued map, the definition of the contingent derivative reduces to the following (see e.g. [1])

$$v_0 \in DF(x_0, y_0)(u_0) \ \Leftrightarrow \liminf_{h \to 0^+} \ dist\left(v_0, \frac{F(x_0 + hu_0) - y_0}{h} \right) = 0.$$

Definition 2. [14] Let X and Y be metric spaces. A set-valued map $F : X \twoheadrightarrow Y$ is *Cellina continuously approximable (abbreviated "CCA")* if for every compact subset K of X

(1) $Gr(F\,|_K)$ is compact;

(2) there exists a sequence $\{f_j\}_{j=1}^{\infty}$ of single-valued continuous maps $f_j : K \to Y$ such that $f_j \xrightarrow{gr} F\,|_K$.

We use $CCA(X,Y)$ to denote the set of all CCA set-valued maps from X to Y.

When $f : X \to Y$ is a single-valued map, then f belongs to $CCA(X,Y)$ if and only if f is continuous.

The CCA property of set-valued maps is strongly related to the following definition of directional GDQs.

Definition 3. [14] Let $m, n \in \mathbb{N}$, $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ be a set-valued map, $\overline{x} \in \mathbb{R}^m$, $\overline{y} \in \mathbb{R}^n$, $\overline{y} \in F(\overline{x})$ and let Λ be a nonempty compact subset of $\mathbb{R}^{n\times m}$ (then an element of Λ is an $n \times m$ matrix). Let S be a subset of $\mathbb{R}^m$. We say that Λ is a *generalized differential quotient (GDQ) of F at $(\overline{x}\ ,\overline{y}\ )$ in the direction of S*, and write $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$ if for every positive real number δ there exist U, G such that
1. U is a compact neighborhood of 0 in $\mathbb{R}^m$ and $U \cap S$ is compact;
2. G is a CCA set-valued map from $\overline{x} + U \cap S$ to the δ-neighborhood Λ^δ of Λ in $\mathbb{R}^{n\times m}$;
3. $G(x) \cdot (x - \overline{x}) \subseteq F(x) - \overline{y}$ for every $x - \overline{x} \in U \cap S$.
For $S = \mathbb{R}^m$ we write $\Lambda \in GDQ(F; \overline{x}, \overline{y})$ and say that Λ is a generalized differential quotient of F at $(\overline{x}, \overline{y})$.

Observe that GDQs are not unique. If $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$, then for any compact overset Λ' of Λ also $\Lambda' \in GDQ(F; \overline{x}, \overline{y}; S)$.

We say that $F : \mathbb{R}^m \twoheadrightarrow \mathbb{R}^n$ is GDQ-differentiable at $(\overline{x}, \overline{y})$ in the direction S if there exists at least one $\Lambda \in GDQ(F; \overline{x}, \overline{y}; S)$.

Definition 4. [8] Let F be GDQ-differentiable at $(\overline{x}, \overline{y})$ in the direction of S. A *minimal GDQ* of F at $(\overline{x}, \overline{y})$ in the direction S is a minimal element of the set of all GDQs of F at this point in the same direction (minimal in the sense of inclusions of sets).

3 A comparison of some theorems on viability

In this section we present and compare some theorems on viability of differential inclusions. The first viability result is due to Nagumo. His theorem concerns single-valued continuous time independent f and closed time independent constraints $K(t) \equiv K_0$ and gives a necessary and sufficient condition for the existence of solution to a problem. Namely,

Theorem 1. *(Nagumo, 1942) Let K_0 be a closed subset of $\mathbb{R}^n$ and let $f : K_0 \to \mathbb{R}^n$ be a continuous, bounded map. A necessary and sufficient condition for a differential equation $\dot{x} = f(x)$ to have a viable solution for any initial condition $x_0 \in K_0$ is*

$$\forall\ x \in K_0, \quad f(x) \in T_{K_0}(x).$$

An easy generalization of the Nagumo result, in the case of time independent set-valued map, i.e. $F(t, y) = F(y)$, is the following theorem which can be found in [1].

Theorem 2. *(Aubin, Cellina 1984) Let K_0 be a closed subset of $\mathbb{R}^n$ and let a multifunction $F : K_0 \twoheadrightarrow \mathbb{R}^n$ be bounded, usc with closed convex values. Then the Cauchy problem*

$$\begin{cases} \dot{y} \in F(y), & \text{almost everywhere in } \mathbb{R}_+ \\ y(0) = y_0, & y_0 \in K_0 \end{cases} \tag{3}$$

has a solution on $\mathbb{R}_+$ for every $y_0 \in K_0$ if and only if

$$F(y) \cap T_{K_0}(y) \neq \emptyset \text{ on } K_0.$$

The above theorem generalizes Nagumo's result replacing a differential equation by an inclusion. Thus, if F is single-valued, we get Nagumo Theorem as a consequence. One can see that the tangential condition from Nagumo Theorem

$$\forall\ x \in K_0, \quad f(x) \in T_{K_0}(x)$$

is replaced in the last theorem by the following

$$F(y) \cap T_{K_0}(y) \neq \emptyset \text{ on } K_0$$

(it is due to convex values of F; in non convex case one has to assume $F(y) \subset T_{K_0}(y)$ on K_0, see e.g. [1]). We call the last condition *the tangential condition.* The tangential condition will also appear in other theorems in the sequel (it may be slightly changed) with an exception of our theorem, where we use GDQ theory instead of the contigent derivative.
Going further, one wants to give sufficient conditions guaranteing the existence of a trajectory of an orientor field remaining in time dependent constraints $K(t)$. This leads us to a generalization of Theorem 2 by Dieter Bothe (in 1992) in [3]. He gave sufficient conditions assuming that for all t and all $y \in K(t)$ the contingent derivative $DK(t, y)(1)$ is nonempty and contains $F(t, y)$ for almost all t, where K is left usc and F is measurable with respect to t and usc with respect to y. Under these conditions there exists a viable solution to the Cauchy problem (2).

Theorem 3. *(Bothe, 1992) Let $T = [0, a] \subset \mathbb{R}$ and $K : T \twoheadrightarrow \mathbb{R}^n$, $Do(K) = T$, be left usc set-valued map with closed convex values such that the interior of $K(t)$ is not empty a.e. in T. Let $F : GrK \twoheadrightarrow \mathbb{R}^n$, $Do(F) = GrK$ have closed convex values, $F(\cdot, y)$ is measurable, $F(t, \cdot)$ is usc and $||F(t, y)|| \leq \alpha(t)(1 + |y|)$ on GrK with $\alpha \in L^1(T)$. Finally, let*

$$\begin{array}{ll} (\{1\} \times F(t, y)) \subset T_{GrK}(t, y) & \text{for all } t \in [0, a) \backslash N, y \in K(t) \\ (\{1\} \times \mathbb{R}^n) \subset T_{GrK}(t, y) \neq \emptyset & \text{for all } t \in N, y \in K(t) \end{array} \tag{4}$$

where $N \subset T$ and $m(N) = 0$. Then (2) has a viable solution.

Another work which concerns the same problem is the article, also from 1992, of: Helene Frankowska, Slawomir Plaskacz and Tadeusz Rzezuchowski. Changing some conditions on K and the tangential condition, the authors give sufficient conditions guaranteing that Cauchy problem (2) has a viable solution.

Theorem 4. *(Frankowska, Plaskacz, Rzezuchowski 1992) Let K be a left absolutely continuous multifunction with closed values on $T = [0, a]$, $F : GrK \twoheadrightarrow \mathbb{R}^n$ have closed, convex values and F is such that $||F(t, y)|| \leq \alpha(t)$; for almost all $t \in T$, for every $y \in K(t)$*

$$\forall \beta > 0, \quad DK(t, y)(1) \cap Liminf_{h\to 0^+} \frac{1}{h} \int_t^{t+h} F(s, y + \beta B)ds \neq \emptyset. \tag{5}$$

Then for every $t_0 \in T$ and $y_0 \in K(t_0)$ there exists a solution $y(\cdot)$ to (2) defined on $[t_0, a]$ and satisfying $y(t) \in K(t)$ for every $t \in [t_0, a]$.

The next work which deals with the problem (2) is the theorem presented in [10] by S. Hu and N. Papageorgiou. The authors give sufficient conditions assuming that K is usc and the orientor field F defined on the graph of K is jointly measurable and usc w.r. to y.

Theorem 5. *(Hu, Papageorgiou 1997) Let $K : T \twoheadrightarrow \mathbb{R}^n$ with nonempty closed values be an usc set-valued map such that for almost all $s \in (0, a)$ and for all $y \in K(s)$, there is a continuous map $t \to y(t)$ on $[0, s]$ or $[s, a]$ such that $y(s) = y$, $DK(\cdot, y(\cdot))$ is closed at s. Let $F : GrK \twoheadrightarrow \mathbb{R}^n$ with closed convex nonempty values satisfy*

(i) $F(t, y)$ *is measurable;*

(ii) $\forall\ t_0 \in [0, a]$ *and* $\forall\ y_0 \in K(t_0)$, $\exists\ \alpha \in L^1(t_0, a)$, $\forall\ (t, y) \in GrK$, $||F(t, y)|| \leq \alpha(t)(1 + |y|)$ *and*

(iii) $\forall\ (t_0, y_0) \in GrK$, $\forall\ t \in [t_0, a]$, $y \mapsto\!\!\!\!\rightarrow F(t, y)$ *is usc.*

Finally, let for every $(t_0, y_0) \in GrK$, for almost every $t \in [t_0, a]$ and for every $y \in K(t)$,

$$F(t, y) \cap DK(t, y)(1) \neq \emptyset.$$

Then for every $(t_0, y_0) \in GrK$ the multivalued Cauchy problem (2) has a viable solution $y : [t_0, a] \to \mathbb{R}^n$ which is an absolutely continuous function.

All above theorems on viability of differential inclusions use the contingent derivative as a main tool in the tangential condition. The contingent derivative, although simple and intuitive, is a very geometrical idea and force specific graphical notation and language. Our idea was to use for the first time in viability theory another tool of differentiation: generalized differential quotients theory. This theory is a generalization of the classical derivative and one does not need to deal with specific language in order to prove some viability results using GDQs.

Let $SGDQ(K; t, y; \mathbb{R}_+)$ denote the closure of the union of all minimal GDQs of K at $(t, y) \in GrK$ in the direction of $\mathbb{R}_+$

Theorem 6. *(Bartosiewicz, Girejko 2005) Consider the multivalued Cauchy problem (2). Assume that* $K : T \twoheadrightarrow \mathbb{R}^n$*, where* $T = [0, a]$*, is a left u.s.c. multifunction with nonempty closed values such that for all* $(t, y) \in GrK$*, where* $t \in [0, a)$*,* K *is GDQ differentiable at* (t, y) *in the direction of* $\mathbb{R}_+$ *and for every* $\varepsilon > 0$ *there exists* $T_\varepsilon \subseteq T$ *such that* $\lambda(T \backslash T_\varepsilon) < \varepsilon$ *and the map* $(t, y) \mapsto SGDQ(K; t, y; \mathbb{R}_+)$ *is u.s.c. on* $(T_\varepsilon \times \mathbb{R}^n) \cap GrK$*. Let* $F : GrK \twoheadrightarrow \mathbb{R}^n$ *with nonempty closed convex values satisfy*
(a) $\forall\ \gamma(\cdot)$*-measurable* $t \mapsto\!\!\!\!\twoheadrightarrow F(t, \gamma(t))$ *is measurable;*
(b) $y \mapsto\!\!\!\!\twoheadrightarrow F(t, y)$ *is u.s.c. for every* $t \in [0, b]$*;*
(c) $||F(t, y)|| \leq a(t)(1 + ||y||)$ *a.e. on* T *with* $a \in L^1(T)$*.*
Additionally, assume that $F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset$ *for almost every* t*,* $(t, y) \in GrK$*. Then for* $y_0 \in K(t_0)$*, problem (2) has a solution.*

Proof. For the proof see [8]. □

Remark 1. In the above theorem the assumption on K to be GDQ-differentiable at every $(t, y) \in (T \times \mathbb{R}^n) \cap GrK$ is important. Indeed, let $T = [0, 1]$ and $y : T \to \mathbb{R}$ be the Cantor function. Thus y is continuous, nondecreasing, $\dot{y}(t) = 0$ for almost every $t \in T$, $y(T) = T$ and $y(\cdot)$ is not absolutely continuous. Let $K(t) = \{y(t)\}$ and $F(t, y) = \{0\}$. Then the tangential condition is satisfied for all $t \in T \backslash N$, $\lambda(N) = 0$, such that $\dot{y}(t) = 0$. However problem (2) has no solution since $0 \notin K(t)$ for $t > 0$.

The main goal of this section, besides presenting a few viability theorems, is to compare them with our result. We compare only these theorems which deal with time dependent constrain multifunction.

Tangential condition
The main difference between our theorem and others is, as it was mentioned before, that we use GDQs theory instead of the contingent derivative to formulate the tangential condition for the problem (2). Namely,

$$F(t, y) \cap SGDQ(K; t, y; \mathbb{R}_+) \neq \emptyset \text{ for almost every } t, (t, y) \in GrK.$$

In other theorems this condition is formulated as follows. In Theorem 3 there is the strongest tangential condition,

$$\begin{array}{ll} (\{1\} \times F(t, y)) \subset T_{GrK}(t, y) & \text{for all } t \in [0, a) \backslash N, y \in K(t) \\ (\{1\} \times \mathbb{R}^n) \subset T_{GrK}(t, y) \neq \emptyset & \text{for all } t \in N, y \in K(t). \end{array}$$

In Theorem 5 the authors put the "classical" tangential condition,

"*for every* $(t_0, y_0) \in GrK$*, for almost every* $t \in [t_0, a]$ *and for every* $y \in K(t)$*,*

$$F(t, y) \cap DK(t, y)(1) \neq \emptyset."$$

Finally, in Theorem 4 authors assume the following:

"for almost all $t \in T$, for every $y \in K(t)$

$$\forall \beta > 0, \quad DK(t,y)(1) \cap Liminf_{h\to 0^+} \frac{1}{h}\int_t^{t+h} F(s, y+\beta B)ds \neq \emptyset." \qquad (6)$$

Assumptions on K.
There are different assumptions on a constraint multifunction K in theorems presented above. In Theorem 3, the author requires K to be *"left usc set-valued map with closed convex values such that the interior of $K(t)$ is not empty a.e. in T"*, in Theorem 4, K has to be *"a left absolutely continuous multifunction with closed values on $T = [0,a]$"*, in Theorem 5 the authors want K to be *"with nonempty closed values; be an usc set-valued map such that for almost all $s \in (0,a)$ and for all $y \in K(s)$, there is a continuous map $t \to y(t)$ on $[0,s]$ or $[s,a]$ such that $y(s) = y$, $DK(\cdot, y(\cdot))$ is closed at s"* and in ours, K is required to be *"a left u.s.c. multifunction with nonempty closed values such that for all $(t,y) \in GrK$, where $t \in [0,a)$, K is GDQ differentiable at (t,y) in the direction of $\mathbb{R}_+$"*. The weakest continuity assumption is left upper semicontinuity. Indeed, it is obvious that every usc set-valued map is left usc, but the converse is not true as the following example shows.

Example 1. Consider the set-valued map $F : \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined as follows:

$$F(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ [0,1] & \text{if } x > 0 \end{cases}$$

Then it is left usc at 0, but not usc at this point.

One can also observe that,

Remark 2. If a set-valued map $K : T \twoheadrightarrow \mathbb{R}^n$ is left absolutely continuous on T then it is left upper semicontinuous on T. Indeed, it is enough to consider for every $\tau_0 \in T$ one of intervals from the definition of left absolutely continuity instead of the union of intervals, i.e.:

$$\tau_0 - t_0 \leq \delta \;\Rightarrow\; \triangle(K(t_0) \cap P, K(\tau_0)) \leq \epsilon$$

what implies left upper semicontinuity of K at every $\tau_0 \in T$, when $K(t)$ is compact for every $t \in T$. In the case when $K(t)$ is only closed, it is also true, because...????

We also assume that a constraint multifunction K is GDQ-differentiable at every $(t,y) \in GrK$. In [7] we proved the following proposition

Proposition 1. *If K is GDQ-differentiable at (s,y) in the direction $\mathbb{R}_+$, then there exists a map $\gamma : [s, s+\delta] \to \mathbb{R}^n$ such that $\gamma(t) \in K(t)$ for $t \in [s, s+\delta]$, $\gamma(s) = y$ and γ is measurable and continuous at s.*

By the above proposition, if we assume GDQ-differentiability of K at every point (t,y), there exists a measurable map $\gamma(\cdot)$ starting at this point. Hence we do not assume additionally the existence of such a map, as it is done in Theorem 5.

Assumptions on F
In all presented above theorems, as well as in ours, $F(t)$ is closed, convex valued. But there are different hypotheses on measurability and bound of F. In Theorem 3, F is require to be measurable with respect to t, in Theorem 4 there is no measurability assumption on F, in Theorem 5 F has to be jointly measurable and in ours the following is require to be fulfilled: $\forall\ \gamma(\cdot)$-*measurable* $t \mapsto F(t, \gamma(t))$ *is measurable.* The strongest assumption is jointly measurability of F in Theaorem 5 and the weakest is, of course, the lack of any assumption on measurability of F as it is in Theorem 4. The price of making weaker assumptions on measurability of F is the necessity of putting stronger assumptions on bound of F (or on something else). And so, in Theorem 4, one can find the strongest bound hypothesis, $||F(t, y)|| \leq \alpha(t)$ with $\alpha \in L^1(T)$. In all other presented theorems there is a weaker hypothesis on a bound of F, namely $||F(t, y)|| \leq \alpha(t)(1 + ||y||)$ a.e. on T with $\alpha \in L^1(T)$. To see that these two assumptions are not equivalent, let us show the following example:

Example 2. Consider the set-valued map $F : \mathbb{R} \times \mathbb{R} \twoheadrightarrow \mathbb{R}$ defined as follows $F(t, y) = [-\frac{1}{\sqrt{t}}(1 + y), \frac{1}{\sqrt{t}}(1 + y)]$. Then it is easy to see that $||F(t, y)|| \leq \alpha(t)(1 + ||y||)$, where $\alpha(t) = \frac{1}{\sqrt{t}}$ and so $\alpha \in L^1(T)$, but there is no such a map $\alpha \in L^1(T)$ that $||F(t, y)|| \leq \alpha(t)$.

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