Euler's Discretization and Dynamic Equivalence of Nonlinear Control Systems

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Abstract. Euler's discretization transforms a nonlinear continuous-time system into a discrete-time one. It is shown that if two continuous-time systems are dynamically feedback equivalent then their Euler's discretizations are dynamically feedback equivalent. Dynamical equivalence is characterized by isomorphism of differential or difference algebras associated to the systems. These algebras form two categories. Euler's discretization defines a covariant functor from the category of differential algebras to the category of difference algebras.

1 Introduction

In 1992 B. Jakubczyk introduced dynamic equivalence of continuous-time nonlinear control systems [6,7]. Dynamic transformations, that were proposed, depended on derivatives of states and controls. Moreover, with any smooth or analytic control system there was associated a differential algebra, i.e. a function algebra with a differential operator.

The definitions and results obtained by Jakubczyk were carried over to nonlinear discrete-time systems [2]. In this case dynamic transformations depended on past and future values of states and controls. Instead of differential algebra, difference algebra was considered. Similarly as in the continuous-time case it was proved that the difference algebra is the only invariant of dynamic feedback equivalence. The results were global, i.e. the transformations of the systems were defined on the entire state space.

In both cases algebraic objects are in one-to-one correspondence with systems, so systems may be represented by differential or difference algebras. They form two categories, with morphisms preserving differential or difference structures. Similar language was used in [3–5,10]. We study the Euler's discretization of a nonlinear continuous-time system. During this process the equation

$$\dot{x} = f(x, u)$$

* Supported by KBN under the Technical University of Bialystok grant W/IMF/3/99

** Supported by KBN under the Technical University of Bialystok grant W/IMF/1/00

is transformed to

$$x(k+1) = x(k) + hf(x(k), u(k))$$

where h is a fixed step of discretization. On the algebraic level, the differential algebra is transformed to the difference algebra. We prove that this transformation is actually a covariant functor from one category into the other. (For another method of discretization see [1]) As an application we get the following:
If two continuous-time systems are dynamically equivalent then their Euler discretizations are dynamically equivalent. In particular, if a continuous-time system is dynamically linearizable then its Euler discretization is dynamically linearizable.

2 Equivalence of Systems

We compare here dynamic equivalence for continuous-time and discrete-time systems. First we introduce some notation.

By $J(A)$ we will denote a disjoint sum of $J_r(A)$ - the set of all sequences $Z_r = (z(r), z(r+1), \ldots)$, where $z(i) \in A$ and $r \in \mathbb{Z}$. If $A = \mathbb{R}^s$, then the set $J(\mathbb{R}^s)$ will be denoted by $J(s)$ and $J_r(\mathbb{R}^s)$ by $J_r(s)$. Similarly, if $A = \mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_l}$ then $J(\mathbb{R}^{s_1} \times \ldots \times \mathbb{R}^{s_l}) = J(s_1, \ldots, s_l)$. The *shift operator* s_k is the map $s_k : J(A) \rightarrow J(A)$ defined by: $s_k(Z_r) := Y_{r+k}$, where $y(i) = z(i-k)$ for $i \geq r+k$ and $k \in \mathbb{Z}$. The *restriction operator* $c_i : J(A) \rightarrow J(A)$ is given by $c_i(Z_r) := Z_{r+i} = (z(r+i), \ldots)$, $i \geq 0$.

We shall consider real maps defined on $J(A)$ where $A \subset \mathbb{R}^{n_1} \times \ldots \mathbb{R}^{n_l}$. We assume that such maps are shift invariant (so we treat them as functions on $J_0(A)$) and depend on a finite number of elements $z(0), z(1), \ldots, z(q)$ of the sequence $Z \in J(A)$, but q depends on a given function φ. We then say that φ *is of finite order*. A *map* $\phi : J(A) \rightarrow \mathbb{R}^{\tilde{s}}$ is *of finite order* if all the components of ϕ have this property.

Let $A \subset \mathbb{R}^{n_1} \times \ldots \mathbb{R}^{n_l}$, $B \subset \mathbb{R}^{\tilde{n}_1} \times \ldots \mathbb{R}^{\tilde{n}_k}$ and $\gamma : J(A) \rightarrow B$. Then the *extension of* γ is the map $\Gamma : J(A) \rightarrow J(B)$, $\Gamma(Z_r) = \tilde{Z}_r$, where $\tilde{Z}_r(i) = \gamma(c_{i-r}(Z_r))$ for $i \geq r$.

2.1 Continuous-Time Systems

Let us consider an analytic or smooth continuous-time control system defined on $\mathbb{R}^n$:

$$\Sigma_c : \dot{x}(t) = f(x(t), u(t)) \tag{1}$$

where $u(t) \in \mathbb{R}^m$ and $t \in \mathbb{R}$. By a *trajectory* of this system we mean any pair $(x(\cdot), u(\cdot))$ that fulfil (1) on some interval. The set of all trajectories of system Σ_c forms a *behavior* of this system, denoted by $B(\Sigma_c)$ (see [11] for the origin of the concept).

By $Z := Jz = (\frac{d^i z}{dt^i})_{i\geq 0}$ we will denote the *infinite jet extension* of a smooth function $t \mapsto z(t)$. Thus if $z(t) \in \mathbb{R}^s$ then Z is a map with values in $J_0(s)$.

Let us consider two continuous-time control systems:

$$\Sigma_c : \dot{x}(t) = f(x(t), u(t)) \quad \text{and} \quad \tilde{\Sigma}_c : \dot{\tilde{x}}(t) = \tilde{f}(\tilde{x}(t), \tilde{u}(t)$$

where $x(t) \in \mathbb{R}^n$, $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$, $u(t), \tilde{u}(t) \in \mathbb{R}^m$, $t \in \mathbb{R}$. We say that systems Σ_c and $\tilde{\Sigma}_c$ are *dynamically feedback equivalent* [6,7] if there exist transformations:

$$x = \phi(\tilde{X}), \qquad u = \psi(\tilde{X}, \tilde{U}) \tag{2}$$

$$\tilde{x} = \tilde{\phi}(X), \qquad \tilde{u} = \tilde{\psi}(X, U) \tag{3}$$

where $\phi, \tilde{\phi}, \psi, \tilde{\psi}$ are maps of class C^s, $s = \omega$ or $s = \infty$, of finite order, such that the induced maps on pairs $(x(\cdot), u(\cdot))$ and $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ preserve behaviors of the systems and are mutually inverse on these behaviors. System Σ_c is *dynamically (feedback) linearizable* if it is dynamically feedback equivalent with a linear controllable one.

2.2 Discrete-Time Systems

Let us consider now a nonlinear analytic or smooth discrete-time control system defined on $\mathbb{R}^n$:

$$\Sigma_d : x(t+1) = g(x(t), u(t)) \tag{4}$$

where $t \in \mathbb{Z}$ and $u(t) \in \mathbb{R}^m$. Let us consider the following sequences: $X_r = (x(r), x(r+1), \ldots) \in J_r(n)$ and $U_r = (u(r), u(r+1), \ldots) \in J_r(m)$. A *trajectory* of system Σ_d is any pair (X_r, U_r) that satisfies (4) for $k \geq r$. The set of all trajectories of the system Σ_d forms the *behavior* of this system. We denote it by $B(\Sigma_d)$. Moreover $B(\Sigma_d) = \bigcup_{r\in Z} B_r(\Sigma_d)$ where $B_r(\Sigma_d)$ is the set of trajectories starting at instant r.

Let us consider two nonlinear discrete-time systems defined, respectively, on $\mathbb{R}^n$ and $\mathbb{R}^{\tilde{n}}$:

$$\Sigma_d : x(t+1) = g(x(t), u(t)) \quad \text{and} \quad \tilde{\Sigma}_d : \tilde{x}(t+1) = \tilde{g}(\tilde{x}(t), \tilde{u}(t))$$

where $u(t), \tilde{u}(t) \in \mathbb{R}^m$. Let us consider also maps:

$$\phi : J(\tilde{n}) \to \mathbb{R}^n, \ \psi : J(\tilde{n}, m) \to \mathbb{R}^m \quad \text{and} \quad \tilde{\phi} : J(n) \to \mathbb{R}^{\tilde{n}}, \ \tilde{\psi} : J(n, m) \to \mathbb{R}^m$$

The extensions Φ, Ψ, $\tilde{\Phi}$, $\tilde{\Psi}$ of $\phi, \psi, \tilde{\phi}, \tilde{\psi}$ define maps

$$\chi : (\tilde{X}_r, \tilde{U}_r) \mapsto (\Phi(\tilde{X}_r), \Psi(\tilde{X}_r, \tilde{U}_r))$$

and

$$\tilde{\chi} : (X_r, U_r) \mapsto (\tilde{\Phi}(X_r), \tilde{\Psi}(X_r, U_r)).$$

The *systems* Σ_d *and* $\tilde{\Sigma}_d$ *are dynamically feedback equivalent* if there exist maps of finite order (ϕ, ψ) and $(\tilde{\phi}, \tilde{\psi})$ such that

$$\chi(B_r(\tilde{\Sigma}_d)) = B_r(\Sigma_d), \quad \tilde{\chi}(B_r(\Sigma_d)) = B_r(\tilde{\Sigma}_d)$$

and χ and $\tilde{\chi}$ are mutually inverse on the behaviors of systems. In other words, ϕ, ψ, $\tilde{\phi}$ and $\tilde{\psi}$ define transformations of states and controls of both systems of the form:

$$\begin{aligned} x(t) &= \phi(\tilde{x}(t), \ldots, \tilde{x}(t+q)), \\ u(t) &= \psi(\tilde{x}(t), \ldots, \tilde{x}(t+q), \tilde{u}(t), \ldots, \tilde{u}(t+q)) \end{aligned}$$

and

$$\begin{aligned} \tilde{x}(t) &= \tilde{\phi}(x(t), \ldots, x(t+q)), \\ \tilde{u}(t) &= \tilde{\psi}(x(t), \ldots, x(t+q), u(t), \ldots, u(t+q)). \end{aligned}$$

Discrete-time system Σ_d is *dynamically (feedback) linearizable* if is dynamically feedback equivalent to a controllable linear one.

3 Differential and Difference Algebras

A *differential algebra* is a commutative algebra A over $\mathbb{R}$ together with a differential operator $D : A \to A$, i.e. a linear map satisfying the Leibniz rule for product.

A map $\tau : A_1 \to A_2$ is a homomorphism of differential algebras (A_1, D_1) and (A_2, D_2) if it is a homomorphism of algebras and satisfies the condition

$$D_2 \circ \tau = \tau \circ D_1. \tag{5}$$

A *difference algebra* is a a commutative algebra A over $\mathbb{R}$ together with a homomorphism $d : A \to A$. If $(A_1, d_1), (A_2, d_2)$ are two difference algebras then a map $\tau : A_1 \to A_2$ is a homomorphism of difference algebras if it is a homomorphism of algebras and satisfies the condition

$$d_2 \circ \tau = \tau \circ d_1. \tag{6}$$

If τ is a bijective map, then τ is an *isomorphism of differential (difference) algebras.*

Let $\mathcal{U}(n, m)$ denote the algebra of all real functions defined on $\mathbb{R}^n \times J(m)$ that are of finite order and are shift invariant. Therefore we may treat them as functions on $\mathbb{R}^n \times J_0(m)$. We will assume that functions $\varphi \in \mathcal{U}(n,m)$ are of the same class as the dynamics f of the given continuous- or discrete-time system.

By the *differential operator* associated with the system Σ_c we will mean the map $D_{\Sigma_c} : \mathcal{U}(n,m) \to \mathcal{U}(n,m)$ [6]:

$$D_{\Sigma_c} := \sum_{1 \le q \le n} f_q \frac{\partial}{\partial x_q} + \sum_{i,j} u_i^{(j+1)} \frac{\partial}{\partial u_i^{(j)}} \tag{7}$$

where $u_i^{(j)} : \mathbb{R}^n \times J(m) \to \mathbb{R}$, $u_i^{(j)}(x,U) = u_i(j)$, $i = 1,\ldots,m$, $j = 0,1,\ldots$. The second sum is treated as a formal sum.

If $l \ge 2$, then $D^l_{\Sigma_c}\varphi := D_{\Sigma_c}(D^{l-1}_{\Sigma_c}\varphi)$. The algebra $\mathcal{U}(n,m)$ together with the differential operator D_{Σ_c} forms a differential algebra, which we will call *the differential algebra of the system* Σ_c and denote by $(\mathcal{U}(n,m), D_{\Sigma_c})$ or shortly by $\mathcal{U}_{\Sigma_c}$.

By the *difference operator* associated with the system Σ_d we mean the map $d_{\Sigma_d} : \mathcal{U}(n,m) \to \mathcal{U}(n,m)$ defined by:

$$(d_{\Sigma_d}\varphi)(x,U) = \varphi(g(x,u(0)), c_1 U) \tag{8}$$

where $\varphi \in \mathcal{U}(n,m)$ and $U = (u(0), u(1), \ldots)$. If $l \ge 2$, then $d^l_{\Sigma_d}\varphi := d_{\Sigma_d}(d^{l-1}_{\Sigma_d}\varphi)$. The algebra $\mathcal{U}(n,m)$ together with the operator d_{Σ_d} forms a difference algebra. We call it *the difference algebra of the system* Σ_d and denote it by $(\mathcal{U}(n,m), d_{\Sigma_d})$ or shortly by $\mathcal{U}_{\Sigma_d}$.

Observe that each Σ_c or Σ_d system uniquely defines respective differential or difference operator and vice versa.

In [6] it was proved that two analytic (or smooth) systems Σ_c and $\tilde{\Sigma}_c$ are dynamically feedback equivalent if and only if their differential algebras are isomorphic. In [2] this result was carried over to the discrete-time case: two discrete-time systems Σ_d and $\tilde{\Sigma}_d$ are dynamically feedback equivalent if and only if their difference algebras are isomorphic. Both results hold under the following assumptions concerning the right-hand sides of (1) or (4):
A1. For every x and y in $\mathbb{R}^n$ there is at most one $u \in \mathbb{R}^m$ such that $y = f(x,u)$ ($y = g(x,u)$ for discrete time).
A2. For every $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$: $\text{rank}\,\partial f/\partial u(x,u) = m$ ($\text{rank}\,\partial g/\partial u(x,u) = m$ for discrete-time).
A3. The map $(x,u) \mapsto (x, f(x,u))$ ($(x,u) \mapsto (x, g(x,u))$ for discrete time) is proper.

4 Categories of Differential and Difference Algebras

We define the *category* $\mathcal{C}$ *of differential algebras of continuous-time systems* as follows:

1. By the class of objects $\mathrm{Ob}\,\mathcal{C}$ of $\mathcal{C}$ we will mean the class of all differential algebras $\mathcal{U}_{\Sigma_c}$ corresponding to continuous-time systems Σ_c satisfying assumptions A1, A2 and A3.
2. By the set of morphisms of $\mathcal{C}$, denoted by $\mathrm{Mor}\,\mathcal{C}$, will be meant the set of all homomorphisms between differential algebras from $\mathrm{Ob}\,\mathcal{C}$. If $(\mathcal{U}_1, D_1)$, $(\mathcal{U}_2, D_2)$ and $(\mathcal{U}_3, D_3) \in \mathrm{Ob}\,\mathcal{C}$ and $\nu : (\mathcal{U}_1, D_1) \rightarrow (\mathcal{U}_2, D_2)$, $\tau : (\mathcal{U}_2, D_2) \rightarrow (\mathcal{U}_3, D_3)$ are morphisms from $\mathrm{Mor}\,\mathcal{C}$, then $\tau \circ \nu \in \mathrm{Mor}\,\mathcal{C}$, $\tau \circ \nu : (\mathcal{U}_1, D_1) \rightarrow (\mathcal{U}_3, D_3)$, is the *composition* of morphisms. It gives again a morphism in $\mathrm{Mor}\,\mathcal{C}$. The identity map $\mathcal{U} \rightarrow \mathcal{U}$ is the *unit morphism*.

Let us also define a *category $\mathcal{D}$ of difference algebras of discrete-time systems.*
1. The class of objects $\mathrm{Ob}\,\mathcal{D}$ of this category is formed by all difference algebras $\mathcal{U}_{\Sigma_d}$ corresponding to discrete-time systems Σ_d satisfying assumptions A1, A2 and A3.
2. The class of morphisms, denoted by $\mathrm{Mor}\,\mathcal{D}$, consists of all homomorphisms between difference algebras from $\mathrm{Ob}\,\mathcal{D}$. Composition of morphisms and the unit morphism are defined as before.

Let $h > 0$ be fixed. By $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ we will denote a functor from category $\mathcal{C}$ into category $\mathcal{D}$ defined in the following way: $\mathcal{F} := (\mathcal{F}_1, \mathcal{F}_2)$ and $\mathcal{F}_1 : \mathrm{Ob}\,\mathcal{C} \rightarrow \mathrm{Ob}\,\mathcal{D}$, $\mathcal{F}_2 : \mathrm{Mor}\,\mathcal{C} \rightarrow \mathrm{Mor}\,\mathcal{D}$. Moreover $\mathcal{F}_1(\mathcal{U}(n,m)) := \mathcal{U}(n,m)$ and $\mathcal{F}_1 D_{\Sigma_c}$ is a difference operator defined by

$$(\mathcal{F}_1 D_{\Sigma_c}) x^i := (\mathrm{id} + h D_{\Sigma_c}) x^i \tag{9}$$

and

$$(\mathcal{F}_1 D_{\Sigma_c}) \varphi(x, U) := \varphi((\mathcal{F}_1 D_{\Sigma_c} x^1, \ldots, \mathcal{F}_1 D_{\Sigma_c} x^n)(x, U), D_{\Sigma_c} U) \tag{10}$$

for any $\varphi \in \mathcal{U}(n,m)$ where id denotes the identity map on $\mathcal{U}(n,m)$, $x^i : \mathbb{R} \times J(m) \rightarrow \mathbb{R}$, $x^i(x, U) = x_i$, $D_{\Sigma_c} U = c_1 U = (u(1), u(2), \ldots)$.

Since $D_{\Sigma_c} x^i(x, U) = f_i(x, u)$, where $u = u(0)$, the difference operator $d = \mathcal{F}_1 D_{\Sigma_c}$ may be written as

$$d\varphi(x, U) = \varphi(x + hf(x, u), U_1).$$

It corresponds to a discrete-time system Σ_d with $g(x, u) = x + hf(x, u)$. It can be easily checked that Σ_d also satisfies assumptions A1, A2 and A3 (because Σ_c does). This system is the *Euler discretization* of Σ_c with the step h.

Finally, if $\tau : A_1 \rightarrow A_2$ is a homomorphism of differential algebras (A_1, D_1) and (A_2, D_2) from $\mathrm{Ob}\,\mathcal{C}$ then $\mathcal{F}_2(\tau) = \tau$ as the homomorphism of algebras (recall that $\mathcal{F}_1(A_1) = A_1$ and $\mathcal{F}_1(A_2) = A_2$). We have to show that τ is in fact a homomorphism of the difference algebras $(A_1, \mathcal{F}_1 D_1)$ and $(A_2, \mathcal{F}_1 D_2)$.

Lemma 1 *Let $\tau \in \mathrm{Mor}\,\mathcal{C}$, $\tau : (\mathcal{U}(n,m), D_{\Sigma_c}) \rightarrow (\mathcal{U}(\tilde{n},m), D_{\tilde{\Sigma}_c})$. Then $\tau \in \mathrm{Mor}\,\mathcal{D}$, i.e. $\tau : (\mathcal{U}(n,m), \mathcal{F}_1 D_{\Sigma_c}) \rightarrow (\mathcal{U}(\tilde{n},m), \mathcal{F}_1 D_{\tilde{\Sigma}_c})$ is also a homomorphism of difference algebras.*

Proof: From the assumption we have

$$\tau D_{\Sigma_c} = D_{\tilde{\Sigma}_c}\tau.$$

We shall show that

$$\tau(\mathcal{F}_1 D_{\Sigma_c}) = (\mathcal{F}_1 D_{\tilde{\Sigma}_c})\tau.$$

Let $\tilde{\pi}_1$ and $\tilde{\pi}_2$ be the projections $(\tilde{x}, \tilde{U}) \mapsto \tilde{x}$ and $(\tilde{x}, \tilde{U}) \mapsto \tilde{U}$.

It was shown in [6] that the map τ is a pullback, i.e. there is a map $\mu : \mathbb{R}^{\tilde{n}} \times J_0(m) \rightarrow \mathbb{R}^n \times J_0(m)$ such that $\tau = \mu^*$. This implies that τ commutes with substitutions. Indeed, let $F : \mathbb{R}^k \rightarrow \mathbb{R}$. Then

$$\begin{aligned}(\tau \circ F)(\varphi_1, \ldots, \varphi_k) &= \tau(F(\varphi_1, \ldots, \varphi_k))\\ &= \tau F(\varphi_1, \ldots, \varphi_k)\\ &= F(\varphi_1 \circ \mu, \ldots, \varphi_k \circ \mu) \qquad = F(\tau\varphi_1, \ldots, \tau\varphi_k).\end{aligned}$$

We are using this property in the following calculation which proves the required equality.

$$\begin{aligned}(\mathcal{F}_1 D_{\tilde{\Sigma}_c})(\tau\varphi) &= (\tau\varphi)((\mathrm{id} + hD_{\tilde{\Sigma}_c})\tilde{\pi}_1, D_{\tilde{\Sigma}_c}\tilde{\pi}_2)\\ &= \varphi(\tau((\mathrm{id} + hD_{\tilde{\Sigma}_c})\tilde{\pi}_1), \tau(D_{\tilde{\Sigma}_c}\tilde{\pi}_2))\\ &= \varphi((\mathrm{id} + hD_{\Sigma_c}, D_{\Sigma_c})(\tau(\tilde{\pi}_1, \tilde{\pi}_2)))\\ &= ((\mathcal{F}_1 D_{\Sigma_c})\varphi)(\tau(\tilde{\pi}_1, \tilde{\pi}_2)) = \tau(d_{\Sigma_d}\varphi).\Box\end{aligned}$$

Lemma 1 gives the following

Proposition 2 *$\mathcal{F}$ is a covariant functor from the category $\mathcal{C}$ into the category $\mathcal{D}$.* □

Let $\mathcal{D}_E$ denote the subcategory $\mathcal{F}(\mathcal{C})$ of $\mathcal{D}$. It consists of all difference algebras that correspond to Euler discretizations (with fixed step h) of continuous-time systems satisfying assumptions A1, A2 and A3. It is a full subcategory of category $\mathcal{D}$, i.e. for any $A, B \in \mathrm{Ob}\,\mathcal{D}_E$ the set of morphisms from A into B is the same in category $\mathcal{D}_E$ as in $\mathcal{D}$. Thus there exists the inverse functor $\mathcal{G} = \mathcal{F}^{-1}$ from $\mathcal{D}_E$ to $\mathcal{C}$.

5 Discretization, Equivalence and Linearization

Now we may apply the result of the previous section to the problems of dynamic equivalence and dynamic linearization.

Theorem 1. *If two continuous-time systems are dynamically equivalent then their Euler discretizations are dynamically equivalent.*

Proof. If two continuous-time systems are dynamically equivalent then their differential algebras are isomorphic. Functors $\mathcal{F}$ transfers the isomorphism of the differential algebras to an isomorphism of the difference algebras. This means that the Euler discretizations of the continuous-time systems are also dynamically equivalent. □

Corollary 3 *If a continuous-time system is dynamically linearizable then its Euler discretization is dynamically linearizable.*

Proof. This follows from Theorem 1 and the fact that if a linear system is controllable then its Euler discretization (also linear) is controllable. □

A *differential algebra* $(\mathcal{U}, D)$ (respectively *difference algebra* $(\mathcal{U}, d)$) is *free* if ([7,9]) there exist $w_1, \ldots, w_r \in \mathcal{U}$ such that
1. for any function $\nu : \mathbb{R}^k \to \mathbb{R}$ of class C^s holds: $\nu \circ W \equiv 0 \Rightarrow \nu \equiv 0$, where $W = \{D^j w_i\}_{i=1,\ldots,r; j=0,1,\ldots}$ (respectively $W = \{d^j w_i\}_{i=1,\ldots,r; j=0,1,\ldots}$)
2. for $\varphi \in \mathcal{U}$ there exists $k \in \mathbb{N}$ and a function $\nu : \mathbb{R}^k \to \mathbb{R}$ of class C^s such that $\varphi = \nu \circ W$.

Functions $w_1, \ldots, w_r$ are called *free generators* of differential algebra $(\mathcal{U}, D)$ (respectively of difference algebra $(\mathcal{U}, d)$).

In [7] it was proved that the continuous-time control system Σ_c is dynamically feedback linearizable if and only if its differential algebra is free. Moreover in [9] it was proved that discrete-time control system Σ_d is dynamically feedback linearizable if and only if its difference algebra is free.

Example 4 Let us consider a linear controllable system with scalar input (control). We may assume that it is in a Brunovsky canonical form:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= u. \end{aligned}$$

Then x_1 is a free generator of the differential algebra of the system. The succesive derivatives of x_1 using the differential operator of the system yields the remaining coordinates and derivatives of u. Euler discretization of the system takes the form:

$$\begin{aligned} x_1(k+1) &= x_1(k) + hx_2(k) \\ x_2(k+1) &= x_2(k) + hx_3(k) \\ &\vdots \\ x_n(k+1) &= x_n(k) + hu(k). \end{aligned}$$

Here again x_1 is a free generator of the difference algebra.

From Corollary 3 we obtain the following

Corollary 5 *Let consider the system Σ_c and its Euler discretization Σ_d. If $(\mathcal{U}(n,m), D_{\Sigma_c})$ is a free differential algebra then $(\mathcal{U}(n,m), d_{\Sigma_d})$ is a free difference algebra.*

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