Observability of discrete–time infinite–dimensional finitely presented linear systems

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Abstract: Discrete–time infinite–dimensional linear dynamic systems, with one–dimensional output, are studied. They are described by infinite matrices with finite rows. Observability of such systems is investigated. It may be described as the property that one can calculate the value of each state variable by using finitely many rows of observability matrix. Conditions of observability of such systems is given.

Key–words: Infinite–dimensional system, linear discrete–time system, observability.

Regular paper, linear systems, lecture session

[*]Supported by KBN under the Technical University of Białystok grant No. W/IMF/2/99

[†]Supported by KBN under the Technical University of Białystok grant No. W/IMF/1/00

1 Introduction

We study here a particular type of discrete–time infinite–dimensional systems with output. They are defined on space $\mathbb{R}^\infty$, the space of all infinite sequences of real numbers, and described by infinite matrices with finite rows, i.e. rows with finitely many nonzero elements.

In papers [3, 4] were presented infinite–dimensional nonlinear analytic systems with output. They were described by functions depending on a finite number of variables. In mentioned papers the criterion of local observability of such systems was given.

One of the main reasons to study such systems is the concept of infinite dynamic extension of a control system. Such an extension of a continuous–time system appeared in works of Fliess [1], Pomet [5], Jakubczyk [2] and others. By analogy, we may consider a dynamic extension of a discrete–time control systems. Let us consider a linear discrete–time control system with output

$$(\Sigma): \quad \begin{array}{rcl} x(k+1) & = & Ax(k) + Bu(k) \\ y(k) & = & Cx(k) + Du(k), \end{array} \tag{1}$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}$ and $y(k) \in \mathbb{R}^p$. Then A,B,C,D are the matrices of dimensions: $n \times n$, $n \times 1$, $p \times n$, $p \times 1$. We extend the state variables by adding $u(k)$ and all its shifts. In this way we get the infinite dynamic extension of system Σ, denoted Σ_∞. It may be written in the followig form:

$$(\Sigma_\infty): \quad \begin{array}{l} x(k+1) = Ax(k) + Bu(k) \\ u(k+1) = u_1(k) \\ u_1(k+1) = u_2(k) \\ \vdots \\ u_k(k+1) = u_{k+1}(k) \\ \vdots \\ y(k) = Cx(k) + Du(k). \end{array} \tag{2}$$

Let $\mathcal{U} = (u_0, u_1, u_2, \ldots)$ be the infinite sequence of control $u = u_0$ and its shifts and $\mathcal{X} = (x, u_0, u_1, \ldots) = (x, \mathcal{U})$. Then Σ_∞ is described by operators $\tilde{A} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ and $\tilde{C} : \mathbb{R}^\infty \longrightarrow \mathbb{R}^p$

identified with infinite matrices of the forms:

$$\tilde{A} = \begin{pmatrix} A & B & \mathbf{0} & \mathbf{0} & \dots \\ \mathbf{0} & \mathbf{0} & 1 & 0 & \dots \\ \mathbf{0} & \mathbf{0} & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} C & D & \mathbf{0} & \dots \end{pmatrix}.$$

Observe that $\tilde{A}$ and $\tilde{C}$ are matrices with finite rows. Let $\tilde{y}(k) = \tilde{C}\mathcal{X}(k)$ be the output of Σ_∞. Then the output $\tilde{y}(k)$ corresponding to $\mathcal{X}(0)$ may be expressed in the following way:

$$\tilde{y}(k) = \tilde{C}\tilde{A}^k\mathcal{X}(0),$$

where $\tilde{A}^k = \tilde{A}^{k-1}\tilde{A}$ and the matrix $\tilde{C}\tilde{A}^k$ has finite rows too.

Another example where we may see such discrete–time infinite–dimensional systems is discretization of partial differential equations of parabolic type where infinitely many variables correspond to infinitely many discretization points of the state space (e.g. real line).

2 Finitely presented mappings on $\mathbb{R}^\infty$

Let $\Pi_n : \mathbb{R}^\infty \longrightarrow \mathbb{R}^n$ denotes the projection on the first n coordinates, that is if $x = (x_1, x_2, \dots) \in \mathbb{R}^\infty$ then $\Pi_n(x) = (x_1, \dots, x_n)$.

We say that a function $\varphi : \mathbb{R}^\infty \longrightarrow \mathbb{R}$ is *finitely presented on* $\mathbb{R}^\infty$ if there exists $k \in \mathbb{N}$ and a function $\tilde{\varphi} : \mathbb{R}^k \longrightarrow \mathbb{R}$ such that $\varphi = \tilde{\varphi} \circ \Pi_k$.

Remark 2.1. Every linear function on $\mathbb{R}^\infty$ of the form $\varphi(x) = \sum_{i=1}^{\infty} a_i x_i$ is finitely presented on $\mathbb{R}^\infty$. Hence for every φ there exists $n \in \mathbb{N}$ such that $\varphi(x) = \sum_{i=1}^{n} a_i x_i$.

We say that a mapping $A : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ is *finitely presented on* $\mathbb{R}^\infty$ if every component of A is finitely presented on $\mathbb{R}^\infty$.

Remark 2.2. Let $A, B : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ be finitely presented mappings on $\mathbb{R}^\infty$. Then the composition $A \circ B$ is finitely presented too.

Lemma 2.3. *Let $A : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ be a linear mapping finitely presented on $\mathbb{R}^\infty$.*

If A is injective then it has the left inverse of the same type, that is linear and finitely presented on $\mathbb{R}^\infty$. □

Corollary 2.4. *Let us consider an infinite system of linear equations:*

$$f_k(x) = \sum_{n=1}^{\infty} a_{kn} x_n = 0, \ (k \in \mathbb{N}), \tag{3}$$

where for every k only finitely many coefficients $a_{kn} \neq 0$. The system (3) has exactly one solution $x = (0, 0, \ldots)$ if and only if for every $i \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that

$$x_i = \sum_{k=1}^{m} b_{ik} f_k(x)$$

for some $b_{ik} \in \mathbb{R}$.

Example 2.5. Let us consider the following system:

$$\begin{aligned} f_1(x) = x_1 + x_2 + x_3 &= 0 \\ f_2(x) = x_2 + x_3 &= 0 \\ &\vdots \\ f_{2k-1}(x) = x_{2k-1} + x_{2k} + x_{2k+1} &= 0 \\ f_{2k}(x) = x_{2k} + x_{2k+1} &= 0 \\ &\vdots \end{aligned}$$

The above system has unique solution $x = 0$ and we can compute every x_k from finitely many functions f_i. For odd k we have that $x_k = f_k - f_{k+1}$ and for even k: $x_k = f_k - f_{k+1} + f_{k+2}$.

3 Discrete–time systems

Let us consider a discrete–time infinite–dimensional system with one–dimensional output:

$$(\Sigma) : \begin{array}{rcl} x(k+1) & = & Ax(k) \\ y(k) & = & Cx(k), \end{array} \tag{4}$$

where $k \in \mathbb{Z}$, $x(k) \in \mathbb{R}^\infty$, $y(k) \in \mathbb{R}$. We identify operator $A : \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$ with an infinite matrix with finite rows and $C : \mathbb{R}^\infty \longrightarrow \mathbb{R}$ with an infinite row with only finite nonzero elements.

For simplicity, we consider here one–dimensional output, but all the results may be formulated for systems with finite–dimensional output.

Let $(x(0), x(1), \ldots), x(i) \in \mathbb{R}^\infty$, be a trajectory of the dynamics of system Σ. Observe that it is uniquely defined by the initial condition $x(0)$. Then the output $y(k)$ corresponding to the initial condition $x(0)$ may be expressed in the following way:

$$y(k) = CA^k x(0),$$

where CA^k is an infinite row with only finitely many nonzero elements.

We say that $x_1, x_2 \in \mathbb{R}^\infty$ are *indistinguishable* (with respect to Σ) if for all $k \geq 0$:

$$CA^k x_1 = CA^k x_2.$$

Otherwise x_1 and x_2 are *distinguishable.*

We say that Σ is *observable* if every two points are distinguishable.

Let C, A be the matrices of system Σ. Observe that for all $k \geq 0$ the rank of the matrix $\begin{pmatrix} C \\ \vdots \\ CA^k \end{pmatrix}$ is finite. Let $i \in \mathbb{N}$. By e_i we will mean the infinite row with 1 at the i–th position and 0 at other postions. As the operator $e_i : \mathbb{R}^\infty \longrightarrow \mathbb{R}$, for $x \in \mathbb{R}^\infty$: $e_i x = x_i$.

Using corrolary 2.4 we can show the following

Theorem 3.1. *System Σ is observable* $\Leftrightarrow \forall i \in \mathbb{N}\ \exists k \geq 0 :$

$$\operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \end{pmatrix} = \operatorname{rank} \begin{pmatrix} C \\ \vdots \\ CA^k \\ e_i \end{pmatrix}.$$

□

Let $D = (d_{ij}) = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix}$. Then the following theorem gives a sufficient condition for observability.

Theorem 3.2. *If* $\forall k \in \mathbb{N}\ \exists k' \geq k\ \exists i_1, \ldots, i_{n_{k'}} \in \mathbb{N} : \forall i = i_1, \ldots, i_{n_{k'}},\ j > k'$

$$d_{ij} = 0 \text{ and } \operatorname{rank}(d_{ij})_{\substack{i = i_1, \ldots, i_{n_{k'}} \\ j = 1, \ldots, k'}} = k'$$

then Σ *is observable.* □

Theorem 3.3. *If* Σ *is observable then* $\operatorname{rank} D = \infty$. □

4 Examples

Example 4.1. Let us consider the following infinite–dimensional system:

$$\begin{aligned}
&x_1(k+1) = x_2(k) - x_1(k) \\
&x_2(k+1) = x_3(k) - x_2(k) \\
&\vdots \\
&x_n(k+1) = x_{n+1}(k) - x_n(k) \\
&\vdots \\
&y(k) = x_1(k).
\end{aligned} \tag{5}$$

Observe that $D = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \ldots \\ -1 & 1 & 0 & 0 & 0 & \ldots \\ 1 & -2 & 1 & 0 & 0 & \ldots \\ -1 & 3 & -3 & 1 & 0 & \ldots \\ \vdots & & & & & \end{pmatrix}$. And it is easy to notice that the condition from theorem 3.2 is satisfied.

Example 4.2. Let us consider the system described by the matrices: $A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 1 & 0 & \ldots \end{pmatrix}$. Then $\operatorname{rank} D = \operatorname{rank} \begin{pmatrix} C \\ CA \\ \ldots \end{pmatrix} = \infty$ but Σ is not observable (e.g.

$x_1 = (1, 0, \ldots)$ and $x_2 = (0, 0, \ldots)$ are indistinguishable).

Example 4.3. Let the system Σ be in the following form

$$(\Sigma): \quad \begin{array}{rcl} x_{2n-1}(k+1) & = & x_{2n}(k) - x_{2n+2}(k) - x_{2n+3}(k) \\ x_{2n}(k+1) & = & x_{2n+1}(k) + x_{2n+3}(k) + x_{2n+4}(k) + x_{2n+5}(k) \ , \quad n \in \mathbb{N}. \\ y(k) & = & x_1(k) + x_2(k) + x_3(k) \end{array}$$

Then the matrix $D = \begin{pmatrix} C \\ CA \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0\ldots \\ 0 & 0 & 1 & 1 & 1 & 0 & \ldots \\ 0 & 0 & 0 & 1 & 1 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{pmatrix}$. From theorem 3.1 the system Σ is observable and the condition from theorem 3.1 is not satisfied. Moreover the equation $Dx = 0$ is like in example 2.5.

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