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REALIZABILITY OF LINEAR CONTROL SYSTEMS ON TIME SCALES

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Abstract. The realization problem of linear time-invariant control systems is extended to time scales. There is given an algorithm for finding a state-space description of the system given by i/o differential equations on time scale. The obtained results are similar to known results for continuous and discrete-time cases.

Key Words. Time scales, linear control systems, generalized Laplace transform, realization.

1. INTRODUCTION

The problem of realization of the given input-output relation practically means finding a dynamical state-space system with input and output able to reproduce, when initialized at a suitable state, the given i/o behavior. This problem has been studied by many authors both in the continuous-time and discrete-time cases for the linear and nonlinear systems, for example [6, 7, 8, 10]. In the linear case the obtained results are similar or even identical. So, there is a question about possibility of unification of existing theories and extension to other time scales , example for $\mathbb{T} = q^{\mathbb{Z}}$, $\mathbb{T} = 2^{\mathbb{N}^2}$, $\mathbb{T} = \sum_{k=0}^{+\infty}[k, k+1]$, etc. Some interesting results on the field of unification were obtained by Goodwin at al. [3], but only for the times $\mathbb{R}$, $\mathbb{Z}$ and $h\mathbb{Z}$.

The one of the best tools for unification and extension of existing theory is theory of time scales. Its language was created in 1988 by Stefan Hilger [4]. One of the main concepts is the delta derivative, which is a generalization of ordinary (time) derivative. If the time scale is the real line, we get ordinary derivative. In the case of integer numbers, delta derivative of a function is the difference of its values at subsequent points. Thus differential equations as well as difference equations are naturally accommodated into the theory. An inverse operation to differentiation, i.e. integration has been also extended into the time scale domain.

The main goal of this paper is to study the realization problem of time-invariant linear control systems on time scales. We show that the generalized Laplace transform can be used to extend the classical results on realizations to the class of linear systems on time scales.

The paper is organized as follows. In the Section 2 we recall the basic facts from calculus on time scales. Section 3 presents the concept of Laplace transform on time scale. Its idea coincides with ordinary Laplace transform and with Z-transform, but one should remember that usually property about shifting in time scales doesn't hold. In Section 4 there are included the main results. In the first one there is given an algorithm for finding the linear realization on time scale starting from differential i/o relation on the same time scale. The second one gives conditions for minimal realization of linear control systems on time scale $\mathbb{T}$. Both results are similar to classical given in continuous case.

2. CALCULUS ON TIME SCALES

We give here a short introduction to differential calculus on time scales. This is a generalization of the standard differential calculus, on one hand, and the calculus of finite differences, on the other hand. Then we describe the inverse operation—integration. This will allow to solve differential equations on time scales. More material on this subject can be found in [2].
A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of real numbers. The standard cases com-

prise $\mathbb{T}=\mathbb{R}$, $\mathbb{T}=\mathbb{Z}$, $\mathbb{T}=h\mathbb{Z}$ for $h>0$. We assume that $\mathbb{T}$ is a topological space with the relative topology induced from $\mathbb{R}$. For $t\in\mathbb{T}$ we define

- the *forward jump operator* $\sigma:\mathbb{T}\to\mathbb{T}$ by $\sigma(t):=\inf\{s\in\mathbb{T}:s>t\}$;
- the *backward jump operator* $\rho:\mathbb{T}\to\mathbb{T}$ by $\rho(t):=\sup\{s\in\mathbb{T}:s<t\}$;
- the *graininess function* $\mu:\mathbb{T}\to[0,\infty)$ by $\mu(t):=\sigma(t)-t$.

If $\sigma(t)>t$, then t is called *right-scattered*, while if $\sigma(t)<t$ is called *left-scattered*. Of $t<\sup\mathbb{T}$ and $\sigma(t)=t$ then t is called *right-dense*. If $t>\inf\mathbb{T}$ and $\sigma(t)=t$, then t is *left-dense*.
We define also the set $\mathbb{T}^k$ as: $\mathbb{T}^k:=\mathbb{T}\setminus(\rho(\sup\mathbb{T}),\sup\mathbb{T}]$ if $\sup\mathbb{T}<\infty$ and $\mathbb{T}^k:=\mathbb{T}$ if $\sup\mathbb{T}=\infty$. Finally, we will denote $f^\sigma:=f\circ\sigma$ for any function $f:\mathbb{T}\to\mathbb{R}$.

Example 2..1

- If $\mathbb{T}=\mathbb{R}$ then for any $t\in\mathbb{R}$, $\sigma(t)=t=\rho(t)$; the graininess function $\mu(t)\equiv 0$.
- If $\mathbb{T}=\mathbb{Z}$ then for every $t\in\mathbb{Z}$, $\sigma(t)=t+1$, $\rho(t)=t-1$; the graininess function $\mu(t)\equiv 1$.
- Let $q>1$. We define time scale $\mathbb{T}=q^{\mathbb{Z}}:=\{q^k:k\in\mathbb{Z}\}$. Then $\sigma(t)=qt$, $\rho(t)=\frac{t}{q}$ and $\mu(t)=(q-1)t$ for all $t\in\mathbb{T}$.

Definition 2..2 Let $f:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$. *Delta derivative* of f at t, denoted by $f^{\triangle}(t)$, is the real number (provided it exists) with the property that given any ε there is a neighborhood $U=(t-\delta,t+\delta)\cap\mathbb{T}$ (for some $\delta>0$) such that

$$|(f(\sigma(t))-f(s))-f^{\triangle}(t)(\sigma(t)-s)|\leq\varepsilon|\sigma(t)-s|$$

for all $s\in U$. Moreover, we say that f is *delta differentiable* on $\mathbb{T}^k$ provided $f^{\triangle}(t)$ exists for all $t\in\mathbb{T}^k$.

In general the function σ need not be differentiable.

Remark 2..3

- If $\mathbb{T}=\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}$ is delta differentiable at $t\in\mathbb{R}$ iff $f^{\triangle}(t)=\lim\limits_{s\to t}\frac{f(t)-f(s)}{t-s}=f'(t)$, i.e. iff f is differentiable in the ordinary sense at t.
- If $\mathbb{T}=\mathbb{Z}$, then $f:\mathbb{Z}\to\mathbb{R}$ is always delta differentiable at every $t\in\mathbb{Z}$ with $f^{\triangle}(t)=\frac{f(\sigma(t))-f(t)}{\mu(t)}=f(t+1)-f(t)=\triangle f(t)$, where $\triangle$ is the usual forward difference operator defined by the last equation above.
- If $\mathbb{T}=q^{\mathbb{Z}}$, then $f^{\triangle}(t)=\frac{f(qt)-f(t)}{(q-1)t}$ for all $t\in\mathbb{T}-\{0\}$.

Example 2..4 The delta derivative of t^2 is $t+\sigma(t)$. This means that the second delta derivative of t^2 may not exists.

A function $f:\mathbb{T}\to\mathbb{R}$ is called *regulated* provided its right-sided limits exist (finite) at all right-dense points at $\mathbb{T}$ and its left-sided limits exist (finite) at all left-dense points in $\mathbb{T}$. A function $f:\mathbb{T}\to\mathbb{R}$ is called *rd-continuous* provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. It can be shown that
- f is continuous $\Rightarrow$ f is rd-continuous $\Rightarrow$ f is regulated
- σ is rd-continuous.

A continuous function $f:\mathbb{T}\to\mathbb{R}$ is called *pre-differentiable* with (the region of differentiation) D, provided $D\subset\mathbb{T}^k$, $\mathbb{T}^k\setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each $t\in D$. It can be proved that if f is regulated then there exists a function F that is pre-differentiable with region of differentiation D such that $F^{\triangle}(t)=f(t)$ for all $t\in D$. Any such function is called pre-antiderivative of f. Then *indefinite integral* of f is defined by $\int f(t)\triangle t:=F(t)+C$ where C is an arbitrary constant. *Cauchy integral* is

$$\int\limits_r^s f(t)\triangle t=F(s)-F(r)\quad\text{for all}\quad r,s\in\mathbb{T}^k$$

A function $F:\mathbb{T}\to\mathbb{R}$ is called an *antiderivative* of $f:\mathbb{T}\to\mathbb{R}$ provided $F^{\triangle}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$.

Remark 2..5 It can be shown that every rd-continuous function has an antiderivative. Moreover, if $f(t)\geq 0$ for all $a\leq t<b$ and $\int\limits_a^b f(\tau)\triangle\tau=0$ then $f\equiv 0$.

Example 2..6
- If $\mathbb{T}=\mathbb{R}$, then $\int\limits_a^b f(\tau)\triangle\tau=\int\limits_a^b f(\tau)d\tau$, where the integral on the right is the usual Riemann integral.
- If $\mathbb{T}=h\mathbb{Z}$, $h>0$, then $\int\limits_a^b f(\tau)\triangle\tau=\sum\limits_{t=\frac{a}{h}}^{\frac{b}{h}-1} f(th)h$ for $a<b$.

Remark 2..7 An antiderivative of 0 is 1, an antiderivative of 1 is t, but it is not possible to find a closed formula of an antiderivative of t: antiderivative of $\frac{t^2}{2}$ is $\frac{t+\sigma(t)}{2}=t+\frac{\mu(t)}{2}$.

Under assumptions that: $a\in\mathbb{T}$, $\sup\mathbb{T}=\infty$ and f is rd-continuous function on $[a,\infty]$ we define *improper integral* by

$$\int\limits_a^\infty f(t)\triangle\tau:=\lim_{b\to\infty}\int_a^b f(t)\triangle\tau$$

provided this limit exists.

3. LAPLACE TRANSFORM

First we make some preliminary definitions. For $h > 0$.

Let $Z_h := \{z \in \mathbb{C} : -\frac{\pi}{h} < \mathrm{Im} z \le \frac{\pi}{h}\}$ and for $h = 0$ let $Z_0 := \mathbb{C}$. Then, let us define a transformation $\xi_h : \{z \in \mathbb{C} : z \neq -\frac{1}{h}\} \to Z_h$ by $\xi_h = \frac{\mathrm{Log}(1+zh)}{h}$, where Log means the principal logarithm function. For $h = 0$ we put $\xi_0(z) := z$ for all $z \in \mathbb{C}$.

We say that function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. For any regressive function p let $(\ominus p)(t) := -\frac{p(t)}{1+\mu(t)p(t)}$.

Definition 3..1 If $p : \mathbb{T} \to \mathbb{R}$ is a regressive function, then the *(generalized) exponential function is defined by*

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(t)}(p(\tau))\triangle\tau\right) \quad \text{for all } s, t \in \mathbb{T}$$

Example 3..2 Let α be any complex constant regressive function, i.e. $\alpha \in \mathbb{C} - \{\frac{1}{h}\}$.

- If $\mathbb{T} = \mathbb{R}$, then $e_\alpha(t, t_0) = e^{\alpha(t-t_0)}$.
- If $h > 0$ and $\mathbb{T} = h\mathbb{Z}$, then $e_\alpha(t, t_0) = (1 + \alpha h)^{\frac{t-t_0}{h}}$.
- If $\mathbb{T} = q^{\mathbb{Z}}$, then $e_\alpha(t, t_0) = \prod_{s\in[t_0,t)} [1 + (q - 1)\alpha s]$ for $t > t_0$.
- If $\mathbb{T} = \{n^2 : n \in \mathbb{N}_0\}$. Then $e_1(t, 0) = 2^{\sqrt{t}}(\sqrt{t})!$.

Theorem 3..3 *Let p be regressive function and fix $t_0 \in \mathbb{T}$. Then $e_p(\cdot, t_0)$ is a unique solution of the initial value problem $x^\triangle = p(t)x$, $x(t_0) = 1$ on $\mathbb{T}$.*

Let us assume that time scale $\mathbb{T}_0$ is such that $0 \in \mathbb{T}_0$ and $\sup \mathbb{T}_0 = \infty$. From now on we will assume that $z = z(t)$ is a constant regressive function.

Definition 3..4 Assume that $x : \mathbb{T}_0 \to \mathbb{R}$ is regulated. The *Laplace transform* of x is defined as

$$\mathcal{L}\{x\}(z) := \int_0^\infty x(t)e^\sigma_{\ominus z}(t, 0)\triangle t$$

for $z \in D\{x\}$, where $D\{x\}$ is the set of all complex regressive constant functions for which the improper integral exists.

It can be proved that Laplace transform has the linearity property. Moreover, one can show that for all regressive constant $z \in \mathbb{C}$

- $\mathcal{L}\{1\}(z) = \frac{1}{z}$ provided $\lim_{t\to\infty} e_{\ominus z}(t, 0) = 0$ holds
- $\mathcal{L}\{x^\triangle\}(z) = zL\{x\}(z) - x(0)$ and $\mathcal{L}\{x^{\triangle\triangle}\}(z) = z^2L\{x\}(z) - zx(0) - x^\triangle(0)$ provided $\lim_{t\to\infty} x(t)e_{\ominus z}(t, 0) = 0$ holds
- $\mathcal{L}\{e_\alpha(\cdot, 0)\}(z) = \frac{1}{z-\alpha}$ provided $\lim_{t\to\infty} e_{\alpha\ominus z}(t, 0) = 0$ holds, where $\alpha \in \mathbb{C}$ is regressive.
- If $\mathbb{T}_0$ has constant $\mu(t) \equiv h \ge 0$, then $\mathcal{L}\{x\}(z) = (1 + hz)L\{x\}(z) - h(1 + hz)x(0)$.

Example 3..5

- If $\mathbb{T}_0 = [0, \infty)$, then the Laplace transform defined above coincides with the standard $\mathcal{L}$-transformation for continuous-time case.
- If $\mathbb{T}_0 = \mathbb{N}_0$ then $(z+1)\mathcal{L}\{x\}(z) = \mathcal{Z}\{x\}(z+1)$ where $\mathcal{Z}\{x\}$ is the usual $\mathcal{Z}$-transform of x for discrete-time case.

Remark 3..6 In general the standard formula for the Laplace transform of the shifted function does not hold.

4. SYSTEMS OF DIFFERENTIAL EQUATIONS

We shall consider here systems of linear differential equations with constant coefficients, defined on time scales.

An $n \times n$ matrix A is called *regressive* with respect to $\mathbb{T}$ provided $I + \mu(t)A$ is invertible for all $t \in \mathbb{T}^k$ (I denotes the identity matrix). The system of delta differential equations $x^\triangle = Ax$ is called *regressive* provided A is regressive.

Remark 4..1 We have the following properties:

- If $\mathbb{T} = \mathbb{R}$, then any matrix A satisfies the regressivity condition.
- If $\mathbb{T} = \mathbb{Z}$, then A is regressive if and only if the matrix $I + A$ is invertible, which holds if and only if -1 is not an eigenvalue of A. In this case the equation $x^\triangle = Ax$ is equivalent to $x(t + 1) = (I + A)x(t)$. Regressivity means that the last equation can be solved backwards.

Theorem 4..2 [2] *Let A be a regressive $n \times n$ matrix. Then the initial value problem*

$$x^\triangle = Ax, \;\; x(t_0) = x_0$$

has a unique solution x defined on $\mathbb{T}$.

Let $t_0 \in \mathbb{T}$ and let A be regressive. The unique matrix-valued solution of the initial value problem $X^\triangle = AX$, $X(t_0) = I$, is called the *matrix exponential function of* A (at t_0). Its value at $t \in \mathbb{T}$ will be denoted by $e_A(t, t_0)$

Remark 4..3 Let A be an $n \times n$ matrix.

- If $\mathbb{T} = \mathbb{R}$, then $e_A(t, t_0) = e^{A(t-t_0)}$.

- If $\mathbb{T} = \mathbb{Z}$ and $I + A$ is invertible, then $e_A(t, t_0) = (I + A)^{(t-t_0)}$.
- If $\mathbb{T} = 2^{\mathbb{N}_0}$ and A is a regressive constant matrix, then $e_A(t, 1) = \prod_{s\in\mathbb{T}\cap(0,t)} (I + sA)$ ordered as: $(I = \rho(t)A)\ldots(I + 2A)(I + A)$.

It can be proved that

1. $e_0(t, s) = I$ and $e_A(t, t) = I$ for every $t, s \in \mathbb{T}$;
2. $e_A(\sigma(t), s) = (I + \mu(t)A)e_A(t, s)$;
3. $e_A(t, s) = e_A^{-1}(s, t)$;
4. $e_A(t, s)e_A(s, r) = e_A(t, r)$.

If A is not regressive we can still compute exponential matrix for $t > t_0$.

Example 4..4 Let us consider linear control system $x^{\triangle}(t) = Ax(t)$ with $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$. Observe that A is not regressive. Let us take $\mathbb{T} := \bigcup_{k\in\mathbb{Z}_+} [2k, 2k+1]$. Then

$$e_A(t, 0) = e^{A(t)}(I + A)^k = (\frac{1}{2}e^{-t}\begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} - \frac{1}{2}e^{-3t}\begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix})\cdot \frac{1}{2}(-2)^k\begin{bmatrix} -1 & -1 \\ 3 & 3 \end{bmatrix}$$

for any $t \in [2k, 2k + 1]$, $k \in \mathbb{Z}_+$.

Let us consider the nonhomogenous equation

$$x^{\triangle}(t) = Ax(t) + f(t), \quad x(t_0) = x_0 \tag{1}$$

where $f : \mathbb{T}\to\mathbb{R}^n$ is a vector-valued rd-continuous function, $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$.

Theorem 4..5 *Let A be a regressive $n \times n$ matrix. Then the initial value problem (1) has a unique solution, given by*

$$x(t) = e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\triangle\tau \tag{2}$$

5. Realization problem

Let Λ be a system defined by the equation

$$P(\triangle)y = Q(\triangle)u \tag{3}$$

where $y \in \mathbb{R}^r$, $u \in \mathbb{R}^m$, $\triangle$ is the differentiation operator on the time scale $\mathbb{T}$ (i.e. $\triangle f = f^{\triangle}$), $P(\triangle)$ and $Q(\triangle)$ are, respectively, $r \times r$ and $r \times m$ matrices whose entries are polynomials in operator $\triangle$. We assume that $\det P(\triangle) \not\equiv 0$.

Let us assume that the maximal degree of the entries of P is d_P and the maximal degree of the entries of Q is d_Q. Let $y^{(k)}(0) = 0$ for $k = 0, \ldots, d_P$ and $u^{(k)}(0) = 0$ for $k = 0, \ldots, d_Q$. Applying the Laplace transform on both sides of the input-output equation (3) we obtain

$$P(z)Y(z) = Q(z)U(z) \tag{4}$$

where $Y(z)$ and $U(z)$ are the Laplace transforms of y and u, respectively. As $P(z)$ is invertible for almost all $z \in \mathbb{C}$, we get $Y(z) = P(z)^1Q(z)U(z)$. As usually, $G_\Lambda(z) = P(z)^{-1}Q(z)$
is called *the transfer matrix* of system Λ.

We will concentrate on the following *realization problem*: having the input-output system given by (3) find a dynamical state-space system defined on time scale $\mathbb{T}$:

$$\begin{aligned} x^{\triangle}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + D_0u + D_1u^{\triangle} + \ldots + D_ru^{\triangle^r} \end{aligned} \tag{5}$$

In the natural way this leads to two main questions: 1. When such realization is minimal? and 2. Does such realization exists on any time scale?

Recall that

- system (5) is *controllable* if for any two states $x_0, x_f \in \mathbb{R}^n$ there exist $t_0, t_f \in \mathbb{T}$, $t_f > t_0$, and control $u(t)$, $t \in [t_0, t_f] \cap \mathbb{T}$ such that for $x_0 = x(t_0)$ one has $x(t_f) = x_f$.
- two states $x_1, x_2 \in \mathbb{R}^n$ are *indistinguishable* if for every control u and for every time $t \in \mathrm{dom}\, u = [t_0, t_u]$ the value of the output $y(t)$ corresponding to u is the same for both initial conditions $x(t_0) = x_1$ and $x(t_0) = x_2$. System (5 is *observable* if any two indistinguishable states are equal.
- system (5) is *minimal* if it is both controllable and observable.

From this moment we shall assume that the time scale $\mathbb{T}$ consists of at least n elements, A is regressive and controls u are piecewise rd-continuous.

Theorem 5..1 *[1] The system (5) is controllable if and only if*

$$\mathrm{rank}[B, AB, \ldots, A^{n-1}B] = n.$$

Theorem 5..2 *[1] The system (5) is observable if and only if*

$$\mathrm{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$$

Proposition 5..3 *There exist real matrices* A, B, C *such that* $P^{-1}(z)Q(z) = C[Iz - A]^{-1}B + \sum_{k=0}^{r} E_i z^i$, *i.e. the system* Λ *can be described by state-space equations (5) on time scale* $\mathbb{T}_0$.

Proof is standard (given for example [5, 9]. We recall one of the algorithms for finding and gives an algorithm for finding matrices $A, B, C, D_0, \ldots, D_r$:

1. Let us decompose the matrix $G_\Lambda(z)$ into proper part $G_{\Lambda_p}(z)$ and polynomial part $D(z) = D_0 + D_1 z + \ldots + D_r z^r$.

2. Let us take into account the proper part of the transfer matrix G_Λ and let us find the least common denominators $E_1(z), E_2(z), \ldots, E_m(z)$:

$$E_i(z) := z^{d_i} - \sum_{k=0}^{d_i-1} a_i^k z^k, \quad i = 1, \ldots, m \quad (6)$$

of each column of this matrix. Then

$$G_{\Lambda_p} = N(z)\mathrm{diag}[\frac{1}{E_1(z)}, \ldots, \frac{1}{E_m(z)}] =$$

$$\begin{bmatrix} \frac{N_{11}(z)}{E_1(z)} & \ldots & \frac{N_{1m}(z)}{E_m(z)} \\ \ldots & \ldots & \ldots \\ \frac{N_{p1}(z)}{E_1(z)} & \ldots & \frac{N_{pm}(z)}{E_m(z)} \end{bmatrix}$$

3. (6) implies that $E(z) = \mathrm{diag}[z^{d_1}, \ldots, z^{d_m}] - A'_m Z$, where

4. Let us defime matrices A and B as follows

$$A := \mathrm{diag}[A_1, \ldots, A_m],$$

$$A_i := \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & \ldots & 1 \\ a_{i0} & a_{i1} & a_{i2} & \ldots & a_{id_i-1} \end{bmatrix},$$

$$i = 1, \ldots, m$$

$$B := \mathrm{diag}[b_1, \ldots, b_m],$$

$$b_i := [0, \ldots, o, 1]^T \in \mathbb{R}^{d_i}$$

One can see that $b_i E_i(z) = [Iz - A_i][1, z, \ldots, z^{d_i-1}]^T$ and $BE(z) = [IZ - A]Z$. Then $C[IZ - A]^{-1}B = CZE^{-1}(z) = N(z)E^{-1}(z) = G_{\Lambda_p}$. □

Theorem 5..4 *For any matrix* $A \in M_{n\times n}$, $B \in M_{n\times m}$, $C \in M_{k\times n}$ *there exist a natural number* $\tilde{n}$ *and matrices* $\tilde{A} \in M_{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in M_{\tilde{n}\times m}$, $\tilde{C} \in M_{k\times\tilde{n}}$ *such that* $Ce_A(t,0)B = \tilde{C}e_{\tilde{A}}\tilde{B}(t,0)$ *for any* $t \in [0, +\infty) \cap \mathbb{T}$ *and pair* $(\tilde{A}, \tilde{B})$ *is controllable and pair* $(\tilde{A}, \tilde{C})$ *is observable.*

Proof: The proof is classic. Let us assume that the pair (A, B) isn't controllable and rank$[A|B] = l < n$. It is known that there exists a nonsingular matrix $P \in M_{n\times n}$ the given system can be decomposed on controllable subsytem (A_{11}, B_1) and noncontrolable one. For $t \in [0, +\infty) \cap \mathbb{T}_0$ let

$$S(t) = \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ 0 & S_{22}(t) \end{bmatrix} := \begin{bmatrix} e_{A_{11}}(t,0) & e_{A_{12}}(t,0) \\ 0 & e_{A_{22}}(t,0) \end{bmatrix}.$$

Then

$$CS(t)B = CP^{-1}PS(t)P^{-1}PB$$

$$= [C_1, 0] \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ 0 & S_{22}(t) \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = C_1 S_{11}(t) B_1$$

for some matrix C_1.

If pair (A, C) isn't observable, then in the similar way one can show also that $CS(t)B = C_1 S_{11} B_1$ for any $t \in [0, +\infty) \cap \mathbb{T}_0$ and pair (A_{11}, C_1) is observable. Hence thesis. □

6. CONCLUDING REMARKS

We have generalized realization problem of linear time-invariant control systems defined to any time scale. Based on classic results we have shown how to find the state-space realization of any system given in operator-differential form. The results that we obtained for systems defined on time scales are similar to classical ones for continuous- and discrete-time systems. A further problem is to extend the time-scales approach to realizations of control systems for the nonlinear case.

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