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Families of Germs in Local Observability of Infinite–Dimensional Systems

Dorota Mozyrska *, Zbigniew Bartosiewicz †
Technical University of Białystok
Wiejska 45A
15-351 Białystok, Poland
dorota@katmat.pb.bialystok.pl, bartos@cksr.ac.bialystok.pl

Abstract

Infinite–dimensional dynamic systems with output are studied. Each function involved in the description of the system depends on a finite number of variables only. Local observability of such systems is defined. It is described by using a new structure called family of germs. The condition of local observability of infinite–dimensional systems is given.
Keywords: Nonlinear control system; infinite–dimensional system; local observability; family of germs.

1 Introduction

In this paper we deal with infinite–dimensional systems with an output. They are defined on space $\mathbb{R}^T$, where T is an arbitrary set. With each system there is associated the observation algebra of this system. In our case it will consist of analytic functions, such that each function depends on a finite number of variables from the infinite set of variables, indexed by T. A motivation for studying such systems is the concept of infinite dynamic extension of a control system ([FL, Po]).

Let T be an arbitrary set, finite or infinite. Then $\mathbb{R}^T = \{x : T \to \mathbb{R}\}$. If $T = \{1, \ldots, n\}$, then the space $\mathbb{R}^T$ is identified with $\mathbb{R}^n$. For a subset

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$S \subset T$, finite or infinite, and $x \in \mathbb{R}^T$ we define $x_S := x_{|S} : S \to \mathbb{R}$, the restriction of x to S. Let $S_1 \subset S_2 \subset T$. By $\Pi^{S_2}_{S_1}$ we denote the projection: $\Pi^{S_2}_{S_1} : \mathbb{R}^{S_2} \to \mathbb{R}^{S_1} : x_{S_2} \mapsto x_{S_1}$. By $\left(\frac{3}{2}\right)^T$ we denote the set of all finite subsets of T.

We say that the function $\varphi : \mathbb{R}^T \to \mathbb{R}$ *depends on finite number of variables* if there exists a finite nonempty subset $S \subset T$ and a function $\overline{\varphi} : \mathbb{R}^S \to \mathbb{R}$ such that: $\varphi = \overline{\varphi} \circ \Pi^T_S$. We say that φ is *analytic* if $\overline{\varphi}$ is analytic. Observe that for finite T this definition is also good and in this case $\overline{\varphi} = \varphi$.

We will consider an universal form of dynamic systems defined on $\mathbb{R}^T$, where T is an arbitrary set. Let us consider the system

$$(\Sigma) : \begin{array}{l} \dot{x}_i = f_i(x), \ i \in T \\ y_j = h_j(x), \ j \in A \end{array} , \tag{1}$$

where A is an arbitrary set. If T is infinite we make an important assumption: all functions f_i, h_j depend on a finite number of variables only. In both cases, finite and infinite, these functions are assumed to be analytic.

By the Lie derivatives of an analytic function $\varphi : \mathbb{R}^T \longrightarrow \mathbb{R}$ with respect to the vector field $f = \sum\limits_{i \in T} f_i$, we mean

$$(L_f\varphi)(x) = \sum_{i \in T} f_i(x) \frac{\partial \varphi}{\partial x_i}(x).$$

Observe that this new function depends again on finite number of variables and it is analytic.

By *the observation algebra* of the system Σ, denoted by $\mathcal{H}(\Sigma)$, we mean the smallest subalgebra of the algebra of real analytic functions on $\mathbb{R}^T$ containing all functions h_j and closed under the Lie derivative with respect to f.

We shall define local observability of the system Σ as local observability of its observation algebra $\mathcal{H}(\Sigma)$, that is as local observability of a family of analytic functions. This concept will be introduced in Section 3.

2 Finite–dimensional case

Let first $T = \{1, \dots, n\}$ and $\mathbb{R}^T = \mathbb{R}^n$. Let $\mathcal{F}$ be any family of real analytic functions on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$. We say that $\mathcal{F}$ is *locally observable at* x_0 if the system of equations:

$$\varphi_i(x) = \varphi_i(x_0), \ \varphi_i \in \mathcal{F} \tag{2}$$

has locally around x_0 only trivial solution $x = x_0$. This means that the level set of $\mathcal{F}$ passing through x_0 consists locally (i.e. in a neighborhood of x_0) of one point. To characterize this property we will use the language of germs. Let us notice that $\mathcal{F}$ is locally observable at x_0 if and only if the germ at x_0 of the level set of $\mathcal{F}$ passing through x_0 is equal x_0.

By $\mathcal{O}_x^n$ we denote the algebra of germs of real analytic functions at x, where $x \in \mathbb{R}^n$, and by m_x^n the maximal ideal of $\mathcal{O}_x^n$, consisting of all germs in $\mathcal{O}_x^n$ that vanish at x.

Let I be an ideal of $\mathcal{O}_x^n$. Then $Z(I)$ denotes the zero set–germ of I (at x). If G_x is a set–germ at point $x \in \mathbb{R}^n$, then $J(G_x)$ denotes the ideal of $\mathcal{O}_x^n$ of germs (at x) of real analytic functions that vanish on G_x.

Now let I be an ideal of a ring P. Then the *real radical of* I, denoted by $\sqrt[\mathbb{R}]{I}$, is the set of all elements $a \in P$ for which there is $m \in \mathbb{N}$, $k \in \mathbb{N} \cup \{0\}$ and $b_1, \ldots, b_k \in P$ such that

$$a^{2m} + b_1^2 + \cdots + b_k^2 \in I.$$

Theorem 1. ([Ri]) *Let $x \in \mathbb{R}^n$, I an ideal of $\mathcal{O}_x^n$. Then $J(Z(I)) = \sqrt[\mathbb{R}]{I}$.* □

Let I_{x_0} be the ideal of $\mathcal{O}_{x_0}^n$ generated by the germs at x_0 of those functions from $\mathcal{F}$ that vanish at x_0.

Theorem 2. ([Ba]) *The family $\mathcal{F}$ is locally observable at x_0 iff $\sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}^n$.* □

3 Infinite–dimensional case

Now let T be an arbitrary infinite set and $\mathcal{F}$ be a famliy of real analytic functions on $\mathbb{R}^T$, $x_0 \in \mathbb{R}^T$. In this case, in the definition of local observability of family $\mathcal{F}$ we do not use topology as in finite-dimensional case, because it does not support the following natural idea based on the finite-dimensional case: we want to decide about the local value of variable looking at a finite number of equations (functions) only. We show that there is no topology on $\mathbb{R}^T$ that could suit the above-mentioned idea.

In the first example we use the product topology. A basis of this topology consists of the sets $U = \prod_{t\in T} U_t$, where U_t is the open subset of $\mathbb{R}$ and $U_t = \mathbb{R}$ for almost all t, except for finite number. It is the weakest topology in which projections: $\Pi^T_{\{i\}} : \mathbb{R}^T \longrightarrow \mathbb{R}^{\{i\}}$ are continuous.

In the second example we will use the cube topology. A basis of this topology are the sets $V_{a,b} = \{x \in \mathbb{R}^T : a_i < x_i < b_i,\ i \in T\}$, where $a, b \in \mathbb{R}^T$ and $\forall(i \in T)\ a_i < b_i$. This topology is stronger than the product topology.

In the first case we show that the product topology is too weak and we need at least the cube topology. In the second we will see that the cube topology is to strong for our idea. Hence we deduce that there is no appriopriate topology.

Let $\mathcal{F}$ be any family of real functions on $\mathbb{R}^T$ and $x_0 \in \mathbb{R}^T$. Then by $W(\mathcal{F}, x_0)$ we denote the level set of $\mathcal{F}$ passing through x_0, i.e. the set of all $x \in \mathbb{R}^T$ such that $\forall(\varphi \in \mathcal{F})\ \varphi(x) = \varphi(x_0)$. Of course $W(\mathcal{F}, x_0)$ is nonempty, because at least $x_0 \in W(\mathcal{F}, x_0)$.

Example 1. Let us consider the family of real functions on $\mathbb{R}^n$: $\mathcal{F} = \{x_1(x_1 - a_1), \ldots, x_n(x_n - a_n)\}$ and point $x_0 = 0 \in \mathbb{R}^n$. Observe that in this finite dimensional case the germ at x_0 of $W(\mathcal{F}, x_0)$ is equal x_0.

Now let $T = \mathbb{N}$, $x_0 = 0 \in \mathbb{R}^{\mathbb{N}}$ and on $\mathbb{R}^{\mathbb{N}}$ we consider the product topology. Let $\mathcal{F} = \{x_i(x_i - a_i),\ i \in \mathbb{N}\}$ where $a_i \neq 0$ for all $i \in \mathbb{N}$. Each function in $\mathcal{F}$ depends on finite number of variables. Observe that $W(\mathcal{F}, x_0)$ consists of points $x \in \mathbb{R}^{\mathbb{N}}$, such that $\forall i \in \mathbb{N} : x_i = 0$ or $x_i = a_i$. It is easy to notice that for every neighborhood U (of x_0), in product topology, we have that $W(\mathcal{F}, x_0) \cap U \neq \{x_0\}$. But we would rather see here a point-germ as in the finite-dimensional case. To achieve this we need stronger topology containing cubes of the form $\prod_{i\in\mathbb{N}}(c_i, d_i)$.

Example 2. Let first $\mathcal{F} = \{x_2^2 - x_1, \ldots, x_n^n - x_1\}$ be the famliy of functions on $\mathbb{R}^n$ and $x_0 = 0 \in \mathbb{R}^n$. In this case the germ of $W(\mathcal{F}, x_0)$ is not equal $\{x_0\}$.

Now let $T = \mathbb{N}$, $x_0 = 0 \in \mathbb{R}^{\mathbb{N}}$ and $\mathcal{F} = \{x_i^i - x_1,\ i \in \mathbb{N}\backslash\{1\}\}$. Let us consider the cube topology and the neighborhood of x_0: $V = \prod_{i\in\mathbb{N}}(-\varepsilon, \varepsilon)$, $0 < \varepsilon < 1$. If $x = (x_1, x_2, \ldots) \in W(\mathcal{F}, x_0)$, then $x_n = \sqrt[n]{x_1}$. Let $x \in W(\mathcal{F}, x_0)$ be such that $0 < x_1 < \varepsilon$. Then exists $n_0 > 1$ such that for all $n \geq n_0$ we have that: $x_n > \varepsilon$, (because $\lim_{n\to\infty} \sqrt[n]{x_1} = 1$). Hence $x \notin V$ and $W(\mathcal{F}, x_0) \cap V = \{x_0\}$. In this case the infinite-dimensional counterpart does not agree with the finite dimensional case either. Here the cube topology is too strong.

To characterize local observability of a family of functions on $\mathbb{R}^T$, where T is an arbitrary set, we will use the following structure.

Definition 1. Let Λ be any partially ordered set and let $P_\Lambda : \Lambda \to \left(\frac{3}{2}\right)^T$: $\alpha \mapsto S_\alpha \in \left(\frac{3}{2}\right)^T$ satisfies the conditions:
i) $\forall(\alpha, \beta \in \Lambda)\ \alpha \leq \beta \Rightarrow S_\alpha = P_\Lambda(\alpha) \subset P_\Lambda(\beta) = S_\beta$,
ii) $\forall(S \in \left(\frac{3}{2}\right)^T)\ \exists(\alpha \in \Lambda) : S \subset S_\alpha$.
Let $x_0 \in \mathbb{R}^T$ and for $\alpha \in \Lambda$, A_α be a set–germ at x_{0S_α} at $\mathbb{R}^{S_\alpha}$.
We say that the set $A = \{A_\alpha, \alpha \in \Lambda\}$ is a family of set–germs at x_0 if

$$\forall(\alpha, \beta \in \Lambda)\ \alpha \leq \beta \Rightarrow A_\beta \subset (\Pi^{S_\beta}_{S_\alpha})^{-1}(A_\alpha).$$

The following definition introduces an equivalence relation for such families.

Definition 2. Let $A = \{A_\alpha, \alpha \in \Lambda\}$ and $B = \{B_\beta, \beta \in \Gamma\}$ be families of set–germs at x_0. Then:
i) $A \preccurlyeq B \ \Leftrightarrow\ \forall(\beta \in \Gamma)\ \exists(\alpha \in \Lambda) : S_\beta \subset S_\alpha \ \wedge\ A_\alpha \subset (\Pi^{S_\alpha}_{S_\beta})^{-1}(B_\beta)$,
ii) $A \approx B \Leftrightarrow A \preccurlyeq B \ \wedge\ B \preccurlyeq A$.

The family $\{A_S = x_{0S}, S \in \left(\frac{3}{2}\right)^T\}$, indexed by $\left(\frac{3}{2}\right)^T$, will be denoted by $\{x_0\}$ and called the family of point–germs at x_0.

By $\mathcal{O}^T_{x_0}$ we denote the algebra of germs of analytic functions at $x_0 \in \mathbb{R}^T$. Every such germ is identified with a germ of an analytic function on some $\mathbb{R}^k$. By $m^T_{x_0}$ we denote the ideal in $\mathcal{O}^T_{x_0}$ generated by $(x_i - x_{0i})_{i \in T}$.

Now let $\mathcal{F}$ be a family of functions such that each function depends on a finite number of variables from the infinite set of variables (indexed by T). Let $\varphi_1, \ldots, \varphi_k \in \mathcal{F}$ depend on $x_i, i \in S$. Then $W(\varphi_1, \ldots, \varphi_k)$ denotes the germ at x_{0S} in $\mathbb{R}^S$ of the level set of $\overline{\varphi}_1, \ldots, \overline{\varphi}_k$, where $\overline{\varphi}_i : \mathbb{R}^S \longrightarrow \mathbb{R}$ and $\varphi_i = \overline{\varphi}_i \circ \Pi^T_S$. By $W_{x_0}(\mathcal{F})$ we denote the family of set–germs at x_0 indexed by $\left(\frac{3}{2}\right)^{\mathcal{F}}$. Any element of the family $W_{x_0}(\mathcal{F})$ has the form $W(\varphi_1, \ldots, \varphi_k)$, where $\{\varphi, \ldots, \varphi_k\} \in \left(\frac{3}{2}\right)^{\mathcal{F}}$.

Definition 3. The family $\mathcal{F}$ *is locally observable at* $x_0 \in \mathbb{R}^T$ if $W_{x_0}(\mathcal{F}) \approx \{x_0\}$.

In other words, $\mathcal{F}$ is locally observable at $x_0 \in \mathbb{R}^T$ if for every $S \in \binom{3}{2}^T$ there exist $\varphi_1, \dots, \varphi_k \in \mathcal{F}$ depending on $x_i, i \in S'$ such that $S \subset S'$ and $W(\varphi_1, \dots, \varphi_k) \subset (\Pi_S^{S'})^{-1}(x_{0S})_{x_{0S'}}$.

Let I_{x_0} be the ideal in $\mathcal{O}_{x_0}^T$ generated by functions $\varphi - \varphi(x_0), \varphi \in \mathcal{F}$. Then we may consider the zero set–germ of ideal I_{x_0} as a family of germs at x_0. The following theorem holds [Mo].

Theorem 3. *The family $\mathcal{F}$ is locally observable at x_0 iff* $\sqrt[\mathbb{R}]{I_{x_0}} = m_{x_0}^T$. □

This result is an extension of the corresponding statement in a finite–dimensional case.

Example 3. Let $T = \mathbb{N}$ and $\mathcal{F} = \{\varphi_i(x) = x_{2i-1}^2 + (x_{2i} - x_{2i+1})^2\}, i \in \mathbb{N}\}$, $x_0 = 0$. Then $W(\varphi_1) = \{x_1 = 0, x_2 = x_3\}$, $W(\varphi_1, \varphi_2) = \{x_1 = x_2 = x_3 = 0, x_4 = x_5\}$ and we see that $\forall i \in \mathbb{N}$ we fix $x_i = 0$ using finitely many functions from $\mathcal{F}$. Hence we get that $W_{x_0}(\mathcal{F}) = \{x_0\}$ and $\mathcal{F}$ is locally observable at x_0. On the other hand one can easily find that $\sqrt[\mathbb{R}]{I_0} = m_0^{\mathbb{N}}$.

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